Low-order divergence-free finite element methods in incompressible fluid mechanics

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Outline

1. Introduction: The main idea, and the Navier-Stokes equation.
2. Viscosity-independent estimates for the time-dependent Navier-Stokes equations.
3. The (generalised) Boussinesq problem.
   - The stabilised method: stability and convergence.
   - Numerical results.
4. Concluding remarks.

Introduction: The main idea

The Navier-Stokes equation : Find $(\boldsymbol{u}, p)$ such that

$$\begin{aligned} -\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad & \text{in } \Omega\,, \\ \text{div}\boldsymbol{u} = 0 \qquad & \text{in } \Omega\,, \\ \boldsymbol{u} = \boldsymbol{0} \qquad & \text{on } \partial\Omega\,. \end{aligned}$$

Introduction: The main idea

The weak formulation : Find $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} + \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{u}\, \boldsymbol{v} - \int_\Omega p\, \mathrm{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,,$$

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Main tools :

- The inf-sup condition

$$\inf_{q \in L_0^2(\Omega)} \sup_{\boldsymbol{v} \in H_0^1(\Omega)^d} \frac{\int_\Omega q \,\mathrm{div} \boldsymbol{v}}{\|q\|_{0,\Omega} |\boldsymbol{v}|_{1,\Omega}} \geq \beta > 0\,.$$

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- The form

$$b(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w}) := \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{v}\, \boldsymbol{w}\,,$$

is antisymmetric. This is,

$$b(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{v}) := \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{v}\, \boldsymbol{v} = 0 \qquad \forall\, \boldsymbol{v} \in H_0^1(\Omega)^d\,.$$

Introduction: The main idea

Some previous works :

1. inf-sup stable conforming :
   - Scott-Vogelius (1984): $\mathbb{P}_k \times \mathbb{P}_{k-1}^{disc}$, for $k \geq 4$ (Guzmán-Scott, 2018);
   - Guzmán-Neilan: Rational bubbles (2014).
2. inf-sup stable non-conforming :
   - Crouzeix-Raviart (1973);
   - Mardal, Tai, Winther (2002). (cubic velocities with constant divergence, $\mathbb{P}_0$ pressures)
3. inf-sup stable $H(div, \Omega)$-conforming :
   - $BDM_k \times \mathbb{P}_k^{disc}$: Oyarzúa, Qin, Schotzau (2014); Lube & Schroeder (2018).
   - Guzmán-Neilan: Brinkman problem (2012).
4. Pressure robust :
   - Modifications based on a discrete Helmholtz decomposition (Linke et. al. 2014-....);

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[1] B., F. Valentin, *Consistent local projection stabilized finite element methods*, **SIAM J. Numer. Anal., 48** 1801–1825, (2010).

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An immediate advantage : Stability of the finite element method can be proven without restrictions on the size of the mesh.

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- For all $(\boldsymbol{v}, q_h) \in \boldsymbol{H}_0^1(\Omega) \times \mathbb{P}_0(\Omega)$, the following continuity holds

$$\|\mathscr{L}(\boldsymbol{v}, q_h)\|_{L^p(\Omega)} \leq C \left(|\boldsymbol{v}|_{1,\Omega} + \left\{ \sum_{F \in \mathcal{F}_h} \tau_F \|[\![q_h]\!]\|_{0,F}^2 \right\}^{\frac{1}{2}} \right),$$

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for all $1 \leq p \leq 6$.

- As a corollary, the error can be estimated as follows:

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$$\sum_{K \in \mathcal{T}_h} |\boldsymbol{u} - \mathscr{L}(\boldsymbol{u}_h, p_h)|_{1,K}^2 = \sum_{K \in \mathcal{T}_h} |\mathscr{L}(\boldsymbol{u}, p) - \mathscr{L}(\boldsymbol{u}_h, p_h)|_{1,K}^2$$

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- As a corollary, the error can be estimated as follows:

$$\sum_{K \in \mathcal{T}_h} |\boldsymbol{u} - \mathscr{L}(\boldsymbol{u}_h, p_h)|_{1,K}^2 = \sum_{K \in \mathcal{T}_h} |\mathscr{L}(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)|_{1,K}^2$$

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- As a corollary, the error can be estimated as follows:

$$\sum_{K \in \mathcal{T}_h} |\boldsymbol{u} - \mathscr{L}(\boldsymbol{u}_h, p_h)|_{1,K}^2 \leq C \left(|\boldsymbol{u} - \boldsymbol{u}_h|_{1,\Omega}^2 + \sum_{F \in \mathcal{F}_h} \tau_F \|[\![p - p_h]\!]\|_{0,F}^2 \right).$$

Thus, $\mathscr{L}(\boldsymbol{u}_h, p_h)$ enjoys the same convergence properties of $(\boldsymbol{u}_h, p_h)$.

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Introduction: The main idea, Why?

Let us consider the Navier–Stokes problem:

$$\left\{ \begin{array}{rcl} -\nu\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p & = & \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{u} & = & 0, \\ \boldsymbol{u} & = & \boldsymbol{0}. \end{array} \right.$$

Smooth solution in $\Omega = (0,1)^2$ given by:

$$u_1 = \sin(\pi x)^2 \sin(2\pi y),$$
$$u_2 = -\sin(\pi y)^2 \sin(2\pi x),$$
$$p = x^2 y - 1/6.$$

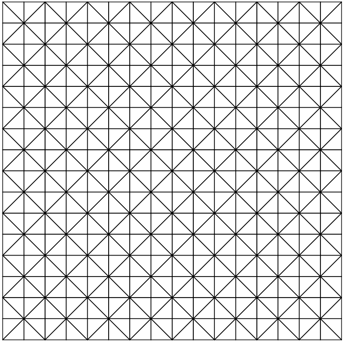
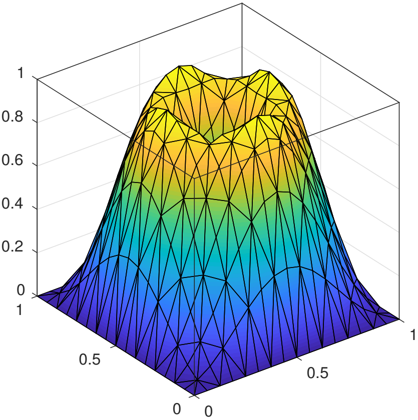
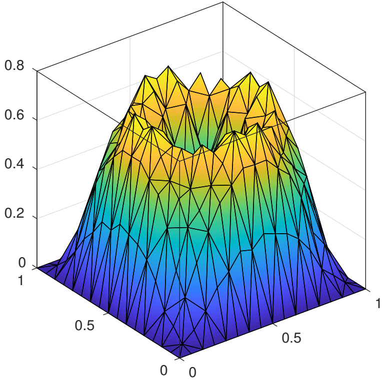


Figure 1: Mesh with 512 elements and 289 vertices.

Taylor–Hood ($\mathbb{P}_2 \times \mathbb{P}_1$) taking $\nu = 10^{-3}$:



Exact $|\boldsymbol{u}|$



Computed $|\boldsymbol{u}_h|$

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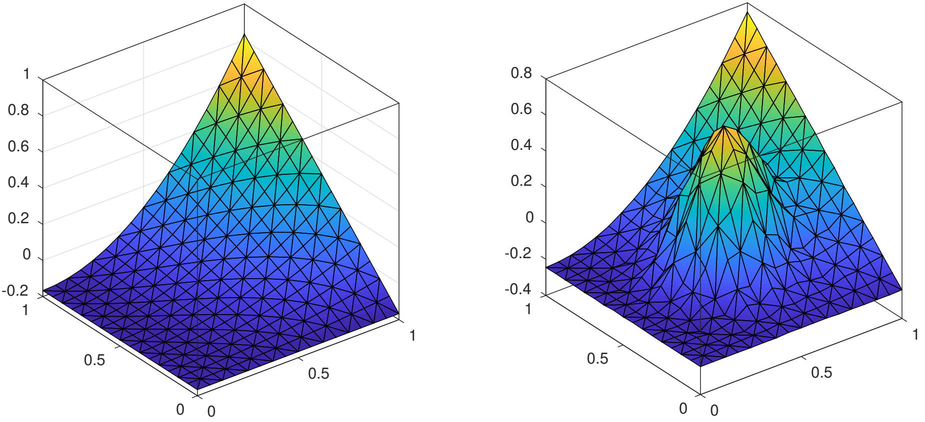
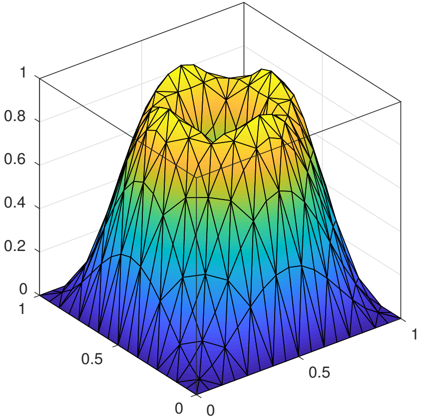


Exact p Computed p_h

Introduction: The main idea, Why?

Taylor–Hood $+\tau\int_\Omega \mathrm{div}\boldsymbol{u}_h \mathrm{div}\boldsymbol{v}_h$ ($\tau = O(1)$, large) taking $\nu = 10^{-3}$:



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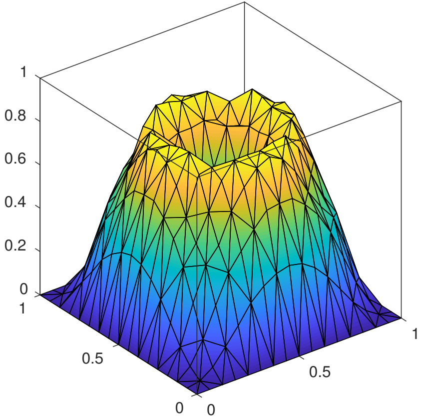

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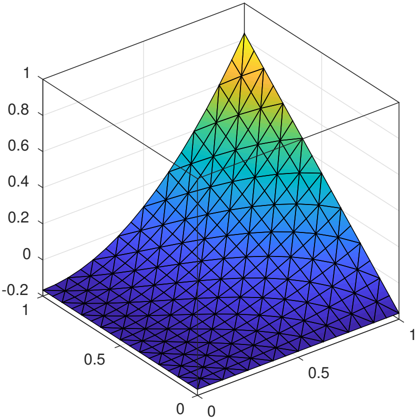
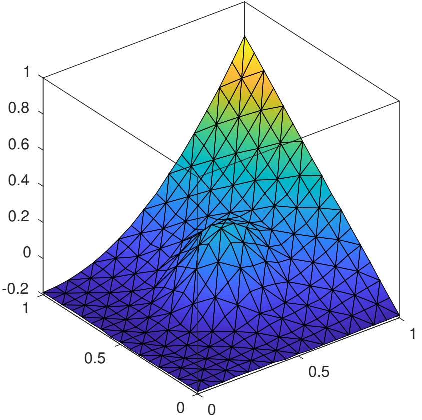
Introduction: The main idea, Why?

Taylor–Hood $+\tau \int_\Omega \mathrm{div}\boldsymbol{u}_h \mathrm{div}\boldsymbol{v}_h$ ($\tau = O(1)$, large) taking $\nu = 10^{-3}$:



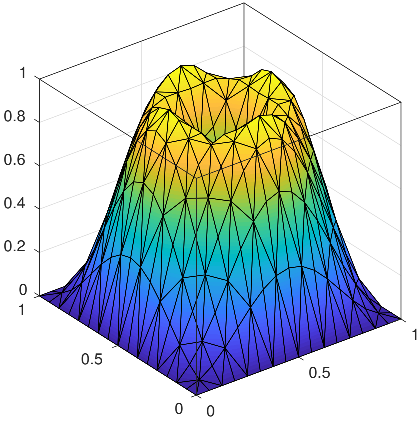
Exact p



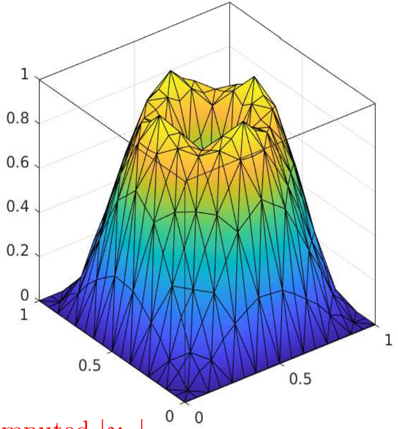
Computed p_h

Introduction: The main idea, Why?

Low-order stabilised $(\mathbb{P}_1 \times \mathbb{P}_0)$ taking $\nu = 10^{-3}$.



Exact $|\boldsymbol{u}|$



Computed $|\boldsymbol{u}_h|$

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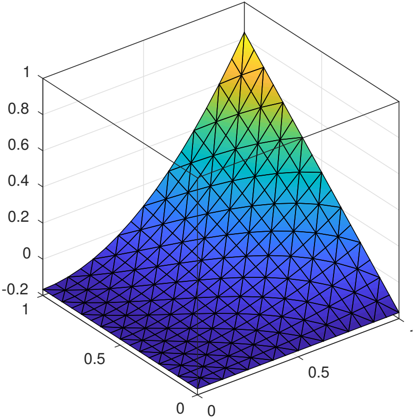
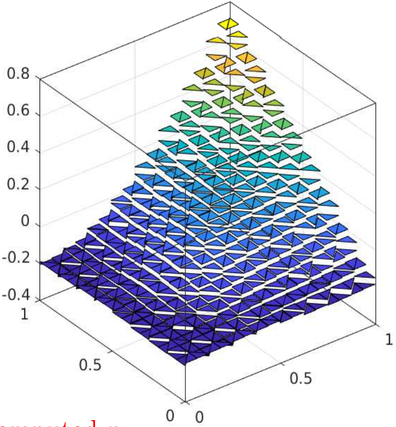
Introduction: The main idea, Why?

Low-order stabilized $(\mathbb{P}_1 \times \mathbb{P}_0)$ taking $\nu = 10^{-3}$:



Exact p



Computed p_h

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Introduction: The main idea, Why?

Computed maximum divergence:

	$\max_{K\in\mathcal{T}_h} \lvert \mathrm{div}\, \boldsymbol{u}_h \rvert$	$\max_{K\in\mathcal{T}_h} \lvert \mathrm{div}\, \mathscr{L}(\boldsymbol{u}_h, p_h) \rvert$
Taylor–Hood	26.7119	-
Taylor–Hood + grad-div	18.3728	-
Low-order stabilized	0.0081	$1.4 \cdot 10^{-14}$

Outline

1. Introduction: The main idea, and the Navier-Stokes equation.
2. Viscosity-independent estimates for the time-dependent Navier-Stokes equations.
3. The (generalised) Boussinesq problem.
   - The stabilised method: stability and convergence.
   - Numerical results.
4. Concluding remarks.

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The time-dependent Navier-Stokes equation

The problem : Find $\boldsymbol{u} : \Omega \times (0,T] \to \mathbb{R}$ and $p : \Omega \times (0,T] \to \mathbb{R}$, such that

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad & \text{in } \Omega \times (0,T]\,, \\ \text{div}\boldsymbol{u} = 0 \qquad & \text{in } \Omega \times (0,T]\,, \\ \boldsymbol{u} = \boldsymbol{0} \qquad & \text{on } \partial\Omega \times (0,T]\,, \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot) \qquad & \text{in } \Omega\,. \end{aligned}$$

Weak problem : Find $(\boldsymbol{u},p) \in L^2((0,T], H_0^1(\Omega)^d) \times L^2((0,T], L_0^2(\Omega))$, such that

$$\begin{aligned} (\partial_t \boldsymbol{u}, \boldsymbol{v})_\Omega + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega + -(p, \text{div}\boldsymbol{v})_\Omega &= (\boldsymbol{f}, \boldsymbol{v})_\Omega\,, \\ (q, \text{div}\boldsymbol{u})_\Omega &= 0\,, \end{aligned}$$

for all $(\boldsymbol{v},q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$, and almost all $t \in (0,T]$, where we have denoted $B(\boldsymbol{u},\boldsymbol{v}) := \boldsymbol{u} \cdot \nabla \boldsymbol{v}$.

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The time-dependent Navier-Stokes equation

The problem : Find $\boldsymbol{u} : \Omega \times (0,T] \to \mathbb{R}$ and $p : \Omega \times (0,T] \to \mathbb{R}$, such that

$$
\begin{aligned}
\partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p &= \boldsymbol{f} \qquad \text{in } \Omega \times (0,T]\,, \\
\mathrm{div}\boldsymbol{u} &= 0 \qquad \text{in } \Omega \times (0,T]\,, \\
\boldsymbol{u} &= \boldsymbol{0} \qquad \text{on } \partial\Omega \times (0,T]\,, \\
\boldsymbol{u}(\cdot,0) &= \boldsymbol{u}_0(\cdot) \qquad \text{in } \Omega\,.
\end{aligned}
$$

Weak problem : Find $(\boldsymbol{u},p) \in L^2((0,T],H_0^1(\Omega)^d) \times L^2((0,T],L_0^2(\Omega))$, such that

$$
\begin{aligned}
(\partial_t \boldsymbol{u}, \boldsymbol{v})_\Omega + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega + (B(\boldsymbol{u},\boldsymbol{u}),\boldsymbol{v})_\Omega - (p, \mathrm{div}\boldsymbol{v})_\Omega &= (\boldsymbol{f},\boldsymbol{v})_\Omega\,, \\
(q, \mathrm{div}\boldsymbol{u})_\Omega &= 0\,,
\end{aligned}
$$

for all $(\boldsymbol{v},q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$, and almost all $t \in (0,T]$, where we have denoted $B(\boldsymbol{u},\boldsymbol{v}) := \boldsymbol{u} \cdot \nabla \boldsymbol{v}$.

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$$(\partial_t \boldsymbol{u},\boldsymbol{v})_\Omega + \nu(\nabla \boldsymbol{u},\nabla \boldsymbol{v})_\Omega + \underbrace{(B(\boldsymbol{u},\boldsymbol{u}),\boldsymbol{v})_\Omega}_{=b(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})} - (p,\operatorname{div}\boldsymbol{v})_\Omega = (\boldsymbol{f},\boldsymbol{v})_\Omega\,,$$

$$(q,\operatorname{div}\boldsymbol{u})_\Omega = 0\,,$$

for all $(\boldsymbol{v},q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$, and almost all $t \in (0,T]$, where we have denoted $B(\boldsymbol{u},\boldsymbol{v}) := \boldsymbol{u} \cdot \nabla \boldsymbol{v}$.

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The time-dependent Navier-Stokes equation

The semi-discrete problem : Find $(\boldsymbol{u}_h, p_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega)$ such that

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h)_\Omega + \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h)_\Omega + b(\mathscr{L}(\boldsymbol{u}_h, p_h); \boldsymbol{u}_h, \boldsymbol{v}_h) - (p_h, \mathrm{div}\boldsymbol{v}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega \,,$$

$$(q_h, \mathrm{div}\boldsymbol{u}_h)_\Omega + \sum_{F \in \mathcal{F}_h} \tau_F([\![p_h]\!], [\![q_h]\!])_F = 0 \,,$$

for almost all $t \in (0, T]$ for all $(\boldsymbol{v}_h, q_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega)$.

The time-dependent Navier-Stokes equation

The semi-discrete problem : Find $(\boldsymbol{u}_h, p_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega)$ such that

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$$(q_h, \mathrm{div}\boldsymbol{u}_h)_\Omega + \underbrace{\sum_{F \in \mathcal{F}_h} \tau_F([\![p_h]\!], [\![q_h]\!])_F}_{= s_{\mathrm{pres}}(p_h, q_h)} = 0 \,,$$

for almost all $t \in (0, T]$ for all $(\boldsymbol{v}_h, q_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega)$.

The time-dependent Navier-Stokes equation

Error estimate : As usual, we split the error as

$$\hat{\boldsymbol{e}}_h := i_h(\boldsymbol{u}) - \boldsymbol{u} \quad , \quad \boldsymbol{e}_h = i_h(\boldsymbol{u}) - \boldsymbol{u}_h \,,$$
$$\hat{\lambda}_h := \Pi_0(p) - p \quad , \quad \lambda_h = \Pi_0(p) - p_h \,.$$

Error equation : Substitute $(\boldsymbol{v}_h, q_h) = (\boldsymbol{e}_h, \lambda_h)$ we get:

$$\begin{aligned}
&(\partial_t \boldsymbol{e}_h, \boldsymbol{v}_h)_\Omega + \nu(\nabla \boldsymbol{e}_h, \nabla \boldsymbol{v}_h)_\Omega + b(i_h(\boldsymbol{u}); i_h(\boldsymbol{u}), \boldsymbol{v}_h) - b(\mathscr{L}(\boldsymbol{u}_h; p_h), \boldsymbol{u}_h, \boldsymbol{v}_h) \\
&- (\lambda_h, \mathrm{div} \boldsymbol{v}_h)_\Omega + (\mathrm{div} \boldsymbol{e}_h, q_h)_\Omega + s_{\mathrm{pres}}(\lambda_h, q_h) \\
&= \big(\partial_t \hat{\boldsymbol{e}}_h, \boldsymbol{v}_h\big)_\Omega + \nu(\nabla \hat{\boldsymbol{e}}_h, \nabla \boldsymbol{v}_h)_\Omega + (\mathrm{div} \hat{\boldsymbol{e}}_h, q_h)_\Omega - b(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(i_h(\boldsymbol{u}); i_h(\boldsymbol{u}), \boldsymbol{v}_h) \\
&+ s_{\mathrm{pres}}(\Pi_0(p), q_h) - (\hat{\lambda}_h, \mathrm{div} \boldsymbol{v}_h)_\Omega \,,
\end{aligned}$$

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The time-dependent Navier-Stokes equation

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Error equation : Substitute $(\boldsymbol{v}_h, q_h) = (\boldsymbol{e}_h, \lambda_h)$ we get:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}_h\|^2_{0,\Omega} + \frac{\nu}{2}\|\nabla \boldsymbol{e}_h\|^2_{0,\Omega} + \frac{1}{2}s_{\text{pres}}(\lambda_h, \lambda_h) \leq$$
$$\|\partial_t \hat{\boldsymbol{e}}_h + B(\mathscr{L}(\boldsymbol{u}_h, p_h), \boldsymbol{u}_h) - B(i_h(\boldsymbol{u}), i_h(\boldsymbol{u}))\|_{0,\Omega}\|\boldsymbol{e}_h\|_{0,\Omega} + C\nu\, h^2|\boldsymbol{u}|_{2,\Omega}$$
$$+ Ch^2\|\boldsymbol{u}\|^2_{2,\Omega} + Ch^2\|p\|_{1,\Omega}$$

The time-dependent Navier-Stokes equation

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$$\|\partial_t \hat{\boldsymbol{e}}_h + B(\mathscr{L}(\boldsymbol{u}_h, p_h), \boldsymbol{u}_h) - B(i_h(\boldsymbol{u}), i_h(\boldsymbol{u}))\|_{0,\Omega}\|\boldsymbol{e}_h\|_{0,\Omega} + C\nu\, h^2|\boldsymbol{u}|_{2,\Omega}$$

$$+ Ch^2\|\boldsymbol{u}\|_{2,\Omega}^2 + Ch^2\|p\|_{1,\Omega}$$

The time-dependent Navier-Stokes equation

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Error equation : Substitute $(\boldsymbol{v}_h, q_h) = (\boldsymbol{e}_h, \lambda_h)$ we get:

$$\frac{d}{dt}\|\boldsymbol{e}_h\|_{0,\Omega}^2 + \nu\|\nabla\boldsymbol{e}_h\|_{0,\Omega}^2 + \frac{5}{4}s_{\text{pres}}(\lambda_h, \lambda_h) \le \left(1 + 2\|\nabla\hat{\boldsymbol{u}}_h\|_{\infty,\Omega} + C\|\hat{\boldsymbol{u}}_h\|_{\infty,\Omega}^2\right)\|\boldsymbol{e}_h\|_0^2$$
$$+ Ch^2(\nu\|\boldsymbol{u}\|_{2,\Omega}^2 + \|p\|_{1,\Omega}^2) + Ch^2\|\boldsymbol{u}_t\|_{1,\Omega}^2 + C\left(\|\boldsymbol{u}\|_{\infty,\Omega}^2 + \|\boldsymbol{u}\|_{2,\Omega}^2\right)h^2\|\boldsymbol{u}\|_{2,\Omega}^2.$$

The time-dependent Navier-Stokes equation

Error estimate : As usual, we split the error as

$$\hat{\boldsymbol{e}}_h := i_h(\boldsymbol{u}) - \boldsymbol{u} \quad , \quad \boldsymbol{e}_h = i_h(\boldsymbol{u}) - \boldsymbol{u}_h \, ,$$
$$\hat{\lambda}_h := \Pi_0(p) - p \quad , \quad \lambda_h = \Pi_0(p) - p_h \, .$$

Error equation : Substitute $(\boldsymbol{v}_h, q_h) = (\boldsymbol{e}_h, \lambda_h)$ we get:

$$\frac{d}{dt}\|\boldsymbol{e}_h\|_{0,\Omega}^2 + \nu\|\nabla \boldsymbol{e}_h\|_{0,\Omega}^2 + \frac{5}{4} s_{\text{pres}}(\lambda_h, \lambda_h) \le \left(1 + 2\|\nabla \hat{\boldsymbol{u}}_h\|_{\infty,\Omega} + C\|\hat{\boldsymbol{u}}_h\|_{\infty,\Omega}^2\right) \|\boldsymbol{e}_h\|_0^2$$
$$+ Ch^2(\nu\|\boldsymbol{u}\|_{2,\Omega}^2 + \|p\|_{1,\Omega}^2) + Ch^2\|\boldsymbol{u}_t\|_{1,\Omega}^2 + C\left(\|\boldsymbol{u}\|_{\infty,\Omega}^2 + \|\boldsymbol{u}\|_{2,\Omega}^2\right) h^2 \|\boldsymbol{u}\|_{2,\Omega}^2 .$$

We multiply the above inequality by $\exp(-K(t))$, where

$$K(t) = \int_0^t \left(1 + C\|\nabla \boldsymbol{u}\|_{\infty,\Omega}^2 + C\|\boldsymbol{u}\|_{\infty,\Omega}^2\right) dr.$$

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The time-dependent Navier-Stokes equation

Theorem

If $(\boldsymbol{u}, p)$ are regular enough, then there exists $C > 0$, independent of h and ν, such that

$$\|\boldsymbol{e}_h(t)\|_{0,\Omega}^2 + \int_0^T \left[\nu \|\nabla \boldsymbol{e}_h(s)\|_{0,\Omega}^2 + s_{\text{pres}}(\lambda_h(s), \lambda_h(s))\right] ds \le C e^{L(T)} h^2 \,,$$

where

$$L(T) = \int_0^T C(1 + \|\nabla \boldsymbol{u}(s)\|_{\infty,\Omega} + \|\boldsymbol{u}(s)\|_{\infty,\Omega}^2) ds \,.$$

The time-dependent Navier-Stokes equation

The fully discrete problem : We introduce $\Delta t = \frac{T}{N}$.

Implicit Euler : given $\boldsymbol{u}_h^0 = i_h(\boldsymbol{u}_0)$, for $i = 0, 1, 2, ...$, find $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ such that

$$\left(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{v}_h\right)_\Omega + \nu(\nabla \boldsymbol{u}_h^{n+1}, \nabla \boldsymbol{v}_h)_\Omega + b(\mathscr{L}(\boldsymbol{u}_h^{n+1}, p_h^{n+1}); \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h)$$

$$-(p_h^{n+1}, \mathrm{div}\boldsymbol{v}_h)_\Omega = (\boldsymbol{f}^{n+1}, \boldsymbol{v}_h)_\Omega$$

$$(\mathrm{div}\boldsymbol{u}_h^{n+1}, q_h)_\Omega + s_{\mathrm{pres}}(p_h^{n+1}, q_h) = 0,$$

for all $(\boldsymbol{v}_h, q_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega)$.

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The time-dependent Navier-Stokes equation

The fully discrete problem : We introduce $\Delta t = \frac{T}{N}$.

Semi-Implicit Euler : for $i = 0, 1, 2, ...,$ find $(\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ such that

$$\left(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{v}_h\right)_\Omega + \nu(\nabla \boldsymbol{u}_h^{n+1}, \nabla \boldsymbol{v}_h)_\Omega + b(\mathscr{L}(\boldsymbol{u}_h^n, p_h^n); \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h)$$
$$-(p_h^{n+1}, \mathrm{div}\boldsymbol{v}_h)_\Omega = (\boldsymbol{f}^{n+1}, \boldsymbol{v}_h)_\Omega$$
$$(\mathrm{div}\boldsymbol{u}_h^{n+1}, q_h)_\Omega + s_{\mathrm{pres}}(p_h^{n+1}, q_h) = 0,$$

for all $(\boldsymbol{v}_h, q_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega)$, where $\boldsymbol{u}_h^0$ solves

$$(\nabla \boldsymbol{u}_h^0, \nabla \boldsymbol{v}_h)_\Omega - (p_h^0, \mathrm{div}\boldsymbol{v}_h)_\Omega + (q_h, \mathrm{div}\boldsymbol{u}_h^0)_\Omega + s_{\mathrm{pres}}(p_h^0, q_h^0) = (\nabla \boldsymbol{u}_0, \nabla \boldsymbol{v}_h)_\Omega \,,$$

for all $(\boldsymbol{v}_h, q_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega)$.

The time-dependent Navier-Stokes equation

The fully discrete problem : We introduce $\Delta t = \frac{T}{N}$.

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for all $(\boldsymbol{v}_h, q_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega)$, where $\boldsymbol{u}_h^0$ solves

$$(\nabla \boldsymbol{u}_h^0, \nabla \boldsymbol{v}_h)_\Omega - (p_h^0, \mathrm{div}\boldsymbol{v}_h)_\Omega + (q_h, \mathrm{div}\boldsymbol{u}_h^0)_\Omega + s_{\mathrm{pres}}(p_h^0, q_h^0) = (\nabla \boldsymbol{u}_0, \nabla \boldsymbol{v}_h)_\Omega\,,$$

for all $(\boldsymbol{v}_h, q_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega)$.

Also, the Crank-Nicolson method has been analysed.

The time-dependent Navier-Stokes equation

Theorem

Let us suppose that

$$\Delta t M_{\boldsymbol{u}} \leq \frac{1}{2} \quad \textit{where} \quad M_{\boldsymbol{u}} = 1 + C\|\nabla \boldsymbol{u}\|_{L^\infty(L^\infty)} + C\|\boldsymbol{u}\|^2_{L^\infty(L^\infty)}\,.$$

Then, there exists $C > 0$, independent of $h, \Delta t$, and ν, such that

$$\begin{aligned}\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|^2_{0,\Omega} &+ \Delta t \nu \sum_{j=1}^{n} \|\nabla(\boldsymbol{u}^j - \boldsymbol{u}_h^j)\|^2_{0,\Omega} + \Delta t \sum_{j=1}^{n} s_{\text{pres}}(p^j - p_h^j, p^j - p_h^j) \\ &\leq C e^{2TM_{\boldsymbol{u}}} \left(\|\boldsymbol{e}_h^0\|^2_{0,\Omega} + TK_{u,p} h^2 + (\Delta t)^2 \|\boldsymbol{u}_{tt}\|_{L^2(L^2)} + Ch^2 \|\boldsymbol{u}_t\|_{L^2(H^1)} \right),\end{aligned}$$

where

$$K_{u,p} = C(\|\boldsymbol{u}\|_{L^\infty(H^2)}, \|\boldsymbol{u}\|_{L^\infty(H^2)}, \|p\|_{L^\infty(H^1)})\,,$$

does not depend on negative powers of ν.

The time-dependent Navier-Stokes equation

Theorem (Error estimate for the pressure)

The following error esimate holds, with a constant independent of $h, \Delta t$, *and* ν:

$$\left\| \Delta t \sum_{j=1}^{n} (p^j - p_h^j) \right\|_{0,\Omega} \leq \beta_0 C(\boldsymbol{u}, \partial_t \boldsymbol{u}, \partial_{tt} \boldsymbol{u}, p, T)(\|\boldsymbol{e}_h\|_{0,\Omega} + h^{2-\frac{d}{2}} + \Delta t).$$ [1]

[1] A. Allendes, B., and J. Novo: *A low-order divergence-free method for the time-dependent Navier-Stokes equation*, in preparation.

Numerical results

We consider the time-space domain $(0,1] \times \Omega$, with $\Omega = (0,1)^2$, and consider as exact solution

$$\boldsymbol{u} = \operatorname{curl}\left(\left(e^t xy(1-x)(1-y)\right)^2\right),$$

$$p = \cos(t)(\sin(x)\cos(y) + (\cos(1)-1)\sin(1)).$$

Numerical results

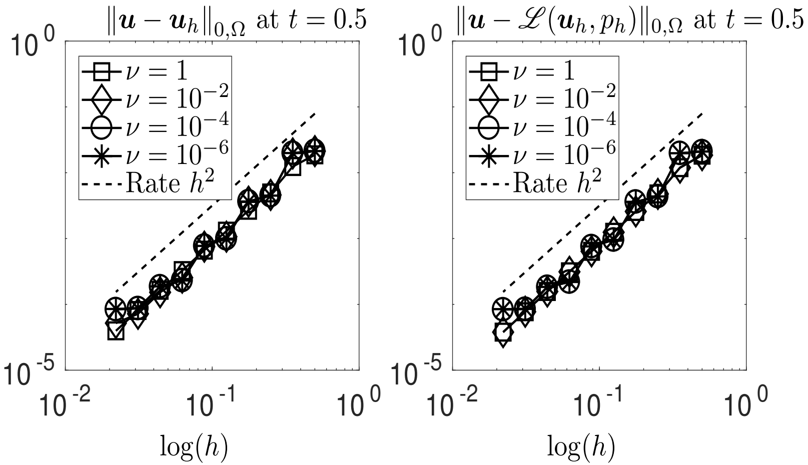

Figure 2: Error for the Implicit Euler Method at time $t = 0.5$. Here $\Delta t = 1/2^{10}$.

Numerical results

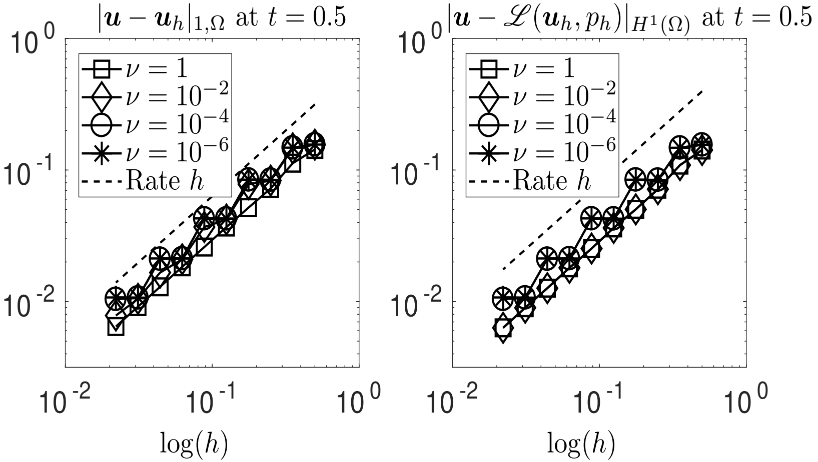

Figure 3: Error for the Implicit Euler Method at time $t = 0.5$. Here $\Delta t = 1/2^{10}$.

Numerical results

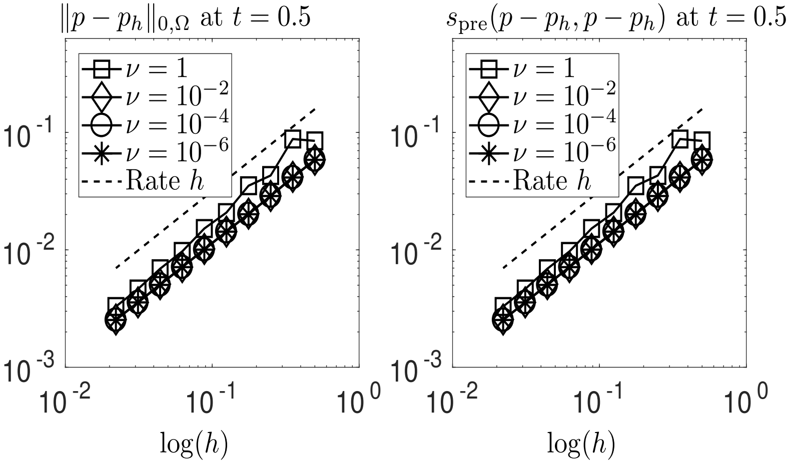

Figure 4: Error for the Implicit Euler Method at time $t = 0.5$. Here $\Delta t = 1/2^{10}$.

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The steady-state Boussinesq problem

The Boussinesq equation :

$$\begin{aligned} -\mathrm{div}(\varepsilon(\mathsf{t})\nabla \boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{g}\mathsf{t} &\quad \text{in } \Omega,\\ \mathrm{div}\ \boldsymbol{u} = 0 &\quad \text{in } \Omega,\\ -\mathrm{div}(\kappa(\mathsf{t})\nabla \mathsf{t}) + \boldsymbol{u}\cdot\nabla\mathsf{t} = 0 &\quad \text{in } \Omega,\\ \boldsymbol{u} = \boldsymbol{0} &\quad \text{on } \Gamma,\\ \mathsf{t} = \mathsf{t}_D &\quad \text{on } \Gamma, \end{aligned}$$

where $\mathsf{t}_D \in H^{\frac{1}{2}}(\Gamma)$, and $\varepsilon(\cdot)$ and $\kappa(\cdot)$ satisfy:

$$0 < \varepsilon_0 \le \varepsilon(\mathsf{t}) \le \varepsilon_1, \quad 0 < \kappa_0 \le \kappa(\mathsf{t}) \le \kappa_1, \quad \forall\, \mathsf{t} \in H^1(\Omega),$$

$$|\varepsilon(\mathsf{t}_1) - \varepsilon(\mathsf{t}_2)| \le C_{\varepsilon,lip}|\mathsf{t}_1 - \mathsf{t}_2|, \quad \forall \mathsf{t}_1, \mathsf{t}_2 \in H^1(\Omega),$$
$$|\kappa(\mathsf{t}_1) - \kappa(\mathsf{t}_2)| \le C_{\kappa,lip}|\mathsf{t}_1 - \mathsf{t}_2|, \quad \forall \mathsf{t}_1, \mathsf{t}_2 \in H^1(\Omega).$$

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The steady-state Boussinesq problem

<u>The weak form</u> : Find $(\boldsymbol{u}, p, \mathsf{t}) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times H^1(\Omega)$ such that:

$$\begin{aligned}
\int_\Omega \varepsilon(\mathsf{t})\nabla\boldsymbol{u} : \nabla\boldsymbol{v} + \int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\boldsymbol{v} - \int_\Omega p\,\mathrm{div}\boldsymbol{v} &= \int_\Omega \mathsf{t}\boldsymbol{g}\cdot\boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\
\int_\Omega q\,\mathrm{div}\boldsymbol{u} &= 0 \quad \forall q \in L_0^2(\Omega), \\
\int_\Omega \kappa(\mathsf{t})\nabla\mathsf{t}\cdot\nabla\psi + \int_\Omega (\boldsymbol{u}\cdot\nabla\mathsf{t})\psi &= 0 \quad \forall \psi \in H_0^1(\Omega), \\
\mathsf{t} &= \mathsf{t}_D \quad \text{on } \Gamma\,.
\end{aligned}$$

The steady-state Boussinesq problem

The well-posedness of the continuous problem :

Theorem

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ *a bounded Lipschitz polyhedral domain. Then, there exists a solution to the weak problem for the Boussinesq equation.*
In addition, if $(\boldsymbol{u}, p, \mathsf{t}) \in \boldsymbol{W}^{1,\infty}(\Omega) \times L^2_0(\Omega) \times W^{1,\infty}(\Omega)$ *satisfies*

$$\max\{\|\boldsymbol{g}\|_{0,\Omega}, \|\boldsymbol{u}\|_{1,\infty,\Omega}, \|\mathsf{t}\|_{1,\infty,\Omega}\} \leq M \, ,$$

where $M > 0$ *is small enough, then the solution is unique.*[2], [3]

[3]S.A. Lorca, J.L. Boldrini, *Stationary solutions for generalized Boussinesq models*, **J. Differential Equations**, **124** (2) 389–406, (1996).

[3]R. Oyarzúa, T. Qin, D. Schötzau, *An exactly divergence-free finite element method for a generalized Boussinesq problem*, **IMA J. Numer. Anal.**, **34** 1104–1135, (2014).

The steady-state Boussinesq problem

Main tools for the proof:

1. The fact that $\boldsymbol{u}$ is solenoidal makes the fixed-point iterates well-defined.
2. Next, $\mathtt{t}$ is decomposed as $\mathtt{t} = \mathtt{t}_0 + \mathtt{t}_1$, with $\mathtt{t}_0 \in H_0^1(\Omega)$ and $\mathtt{t}_1|_\Gamma = \mathtt{t}_D$, and the a priori estimate is shown for each fixed-point iterate:

$$(1-)\,\|\nabla \boldsymbol{u}\|_{0,\Omega} \leq \frac{C\|\boldsymbol{g}\|_{0,\Omega}}{\varepsilon_0\kappa_0}\|\mathtt{t}_1\|_{1,\Omega}(\kappa_0+\kappa_1).$$

Lemma

Let $\mathtt{t}_D \in H^{\frac{1}{2}}(\Gamma)$. Then, for every $\varepsilon > 0$ and $1 \leq p \leq 6$, there exists an extension $\mathtt{t}_1 \in H^1(\Omega)$ of $\mathtt{t}_D$ such that $\|\mathtt{t}_1\|_{L^p(\Omega)} < \varepsilon$.

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$$\left(1 - \frac{C^2}{\varepsilon_0 \kappa_0} \|\boldsymbol{g}\|_{0,\Omega} \|\mathtt{t}_1\|_{L^3(\Omega)}\right) \|\nabla \boldsymbol{u}\|_{0,\Omega} \leq \frac{C\|\boldsymbol{g}\|_{0,\Omega}}{\varepsilon_0 \kappa_0} \|\mathtt{t}_1\|_{1,\Omega} (\kappa_0 + \kappa_1).$$

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Let $\mathtt{t}_D \in H^{\frac{1}{2}}(\Gamma)$. Then, for every $\varepsilon > 0$ and $1 \leq p \leq 6$, there exists an extension $\mathtt{t}_1 \in H^1(\Omega)$ of $\mathtt{t}_D$ such that $\|\mathtt{t}_1\|_{L^p(\Omega)} < \varepsilon$.

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The stabilised finite element method

The stabilised method : Find $(\boldsymbol{u}_h, p_h, \mathsf{t}_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega) \times \mathbb{P}_1(\Omega)$ such that $\mathsf{t}_h|_\Gamma = i_h(\mathsf{t}_D)$, and

$$\int_\Omega \varepsilon(\mathsf{t}_h)\nabla\boldsymbol{u}_h : \nabla\boldsymbol{v}_h + \int_\Omega \boldsymbol{u}_h\cdot\nabla\boldsymbol{u}_h\,\boldsymbol{v}_h + S_u(\boldsymbol{u}_h,\boldsymbol{v}_h) - \int_\Omega p_h \mathrm{div}\boldsymbol{v}_h = \int_\Omega \boldsymbol{g}\mathsf{t}_h\cdot\boldsymbol{v}_h\,,$$

$$\int_\Omega q_h\mathrm{div}\boldsymbol{u}_h + \sum_{F\in\mathcal{F}_h}\tau_F\int_F [\![p_h]\!][\![q_h]\!] = 0\,,$$

$$\int_\Omega \kappa(\mathsf{t}_h)\nabla\mathsf{t}_h\cdot\nabla\psi_h + \int_\Omega \boldsymbol{u}_h\cdot\nabla\mathsf{t}_h\,\psi_h + S_{\mathsf{t}}(\mathsf{t}_h,\psi_h) = 0\,,$$

for all $(\boldsymbol{v}_h, q_h, \psi_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega) \times \mathbb{P}_1(\Omega)$.

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- S_u and S_t are symmetric, and semi-positive definite, tipically, grad-div and CIP, respectively.
- This guarantees that the convective field fed back to the momentum and temperature equations is divergence-free.

The fixed-point algorithm

Initial step : We start with $(\boldsymbol{u}_h^0, p_h^0)$ solution of Stokes, and t_h^0 any initial datum.
For every step : Find $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}, \mathsf{t}_h^{n+1}) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega) \times \mathbb{P}_1(\Omega)$ such that $\mathsf{t}_h^{n+1}|_\Gamma = i_h(\mathsf{t}_D)$, and

$$\int_\Omega \varepsilon(\mathsf{t}_h^n)\nabla\boldsymbol{u}_h^{n+1} : \nabla\boldsymbol{v}_h + \int_\Omega \mathscr{L}(\boldsymbol{u}_h^n, p_h^n) \cdot \nabla\boldsymbol{u}_h^{n+1}\, \boldsymbol{v}_h + S_u(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) - \int_\Omega p_h^{n+1} \mathrm{div}\boldsymbol{v}_h = \int_\Omega \boldsymbol{g}\mathsf{t}_h^n \cdot \boldsymbol{v}_h \,,$$

$$\int_\Omega q_h \mathrm{div}\boldsymbol{u}_h^{n+1} + \sum_{F\in\mathcal{F}_h} \tau_F \int_F [\![p_h^{n+1}]\!][\![q_h]\!] = 0\,,$$

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for all $(\boldsymbol{v}_h, q_h, \psi_h) \in \mathbb{P}_1(\Omega)^d \times \mathbb{P}_0(\Omega) \times \mathbb{P}_1(\Omega)$.

The stabilised finite element method

Remarks on the scheme :

1. On each step of a fixed-point algorithm the linear problem is well-posed, thanks to the hypotheses on ε, κ, and the solenoidal character of $\mathscr{L}(\boldsymbol{u}_h^n, p_h^n)$.
2. The fixed-point mapping is continuous thanks to the properties of ε, κ and $\mathscr{L}(\cdot)$.
3. We split $\mathrm{t}_h = \mathrm{t}_{h,0} + \mathrm{t}_{h,1}$, with $\mathrm{t}_{h,0} \in H_0^1(\Omega) \cap \mathbb{P}_1(\Omega)$ and $\mathrm{t}_{h,1} \in \mathbb{P}_1(\Omega)$ is such that $\mathrm{t}_{h,1}|_\Gamma = i_h(\mathrm{t}_D)$. Mimicking the analysis of the continuous problem we can show that

$$(1-) \, \|\nabla \boldsymbol{u}_h\|_{0,\Omega} \leq \frac{C\|\boldsymbol{g}\|_{0,\Omega}}{\varepsilon_0 \kappa_0} \|\mathrm{t}_{h,1}\|_{1,\Omega} (\kappa_0 + \kappa_1).$$

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Existence result for the discrete scheme

Theorem

Let us assume that the mesh is fine enough, or the boundary datum $\mathtt{t}_D$ is small enough, such that

$$\frac{C^2}{\varepsilon_0 \kappa_0} \|\boldsymbol{g}\|_{0,\Omega} \|\mathtt{t}_{h,1}\|_{L^3(\Omega)} \leq \frac{1}{2} \,.$$

Then,

1. *there exists a solution $(\boldsymbol{u}_h, p_h, \mathtt{t}_h)$ of the discrete scheme.*
2. *the hidden velocity field $\mathscr{L}(\boldsymbol{u}_h, p_h)$ is pointwise divergence-free.*

Error analysis

Further hypotheses on the stabilsing terms :

1. $S_u(\boldsymbol{v},\boldsymbol{w}) = S_u(\boldsymbol{w},\boldsymbol{v})$ and $S_{\mathsf{t}}(\phi,\psi) = S_{\mathsf{t}}(\psi,\phi)$.
2. $S_u(\boldsymbol{v},\boldsymbol{v}) \geq 0$ and $S_{\mathsf{t}}(\psi,\psi) \geq 0$.
3. For all $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega)$ and $\psi \in H^2(\Omega)$ the following holds:

$$S_u(\boldsymbol{v},\boldsymbol{v}) \leq Ch^2\|\boldsymbol{v}\|^2_{2,\Omega} \quad \text{and} \quad S_{\mathsf{t}}(\psi,\psi) \leq Ch^2\|\psi\|^2_{2,\Omega} .$$

4. Examples used in our calculations :

$$S_u(\boldsymbol{v},\boldsymbol{w}) = \begin{cases} \sum_{K\in\mathcal{T}_h} \tau_K \int_K \operatorname{div}\boldsymbol{v}\,\operatorname{div}\boldsymbol{w} & \text{div-div} \\ \sum_{F\in\mathcal{F}_h} \tilde{\tau}_F \int_F [\![\partial_{\boldsymbol{n}}\boldsymbol{v}]\!][\![\partial_{\boldsymbol{n}}\boldsymbol{w}]\!] & \text{CIP} \\ 0 & \text{zero}. \end{cases},$$

$$S_{\mathsf{t}}(\phi,\psi) = \sum_{F\in\mathcal{F}_h} \tilde{\tau}_F \int_F [\![\partial_{\boldsymbol{n}}\phi]\!][\![\partial_{\boldsymbol{n}}\psi]\!] ,$$

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Error analysis

Theorem

Let us suppose that

$$\max\{\|\boldsymbol{g}\|_{0,\Omega}, \|\boldsymbol{u}\|_{1,\infty,\Omega}, \|\mathsf{t}\|_{1,\infty,\Omega}\} \leq M,$$

where M is small enough. Then, the following estimate holds:

$$\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|_{\mathcal{T}_h} + \|\mathsf{t} - \mathsf{t}_h\|_h \leq Ch(\|\boldsymbol{u}\|_{2,\Omega} + \|p\|_{1,\Omega} + \|\mathsf{t}\|_{2,\Omega}).$$

Moreover,

$$\left\{\sum_{K\in\mathcal{T}_h} |\boldsymbol{u} - \mathscr{L}(\boldsymbol{u}_h, p_h)|^2_{1,K}\right\}^{\frac{1}{2}} \leq Ch(\|\boldsymbol{u}\|_{2,\Omega} + \|p\|_{1,\Omega} + \|\mathsf{t}\|_{2,\Omega}).$$

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Error analysis

Bound on the non-conformity error :

Lemma

Let, for every $\gamma \in \mathcal{F} \cap \Gamma$, $\boldsymbol{t}_\gamma$ *be any unit vector parallel to* γ. *Then, there exists* $C > 0$, *independent of* h, *such that*

$$\left(\sum_{\gamma \in \mathcal{F} \cap \Gamma} \|\mathscr{L}(\boldsymbol{u}_h, p_h) \cdot \boldsymbol{t}_\gamma\|^2_{L^2(\gamma)} \right)^{\frac{1}{2}} \leq C h^{\frac{1}{2}} \left(\sum_{\gamma \in \mathcal{F}_I} \tau_\gamma \| [\![p - p_h]\!]_\gamma \|^2_{L^2(\gamma)} \right)^{1/2} .$$ [4]

[4] Allendes, B. and Naranjo: *A divergence-free low-order stabilized finite element method for a generalized steady state Boussinesq problem.* **Comput. Methods Appl. Mech. Engrg. 340**, 90–120 , 2018.

Numerical results

Data: $\varepsilon(\mathsf{t}) = e^{-\mathsf{t}}, \kappa(\mathsf{t}) = e^{\mathsf{t}}$, $\boldsymbol{u}(x,y) = (\sin(\pi y), \cos(\pi x))^T$, $p(x,y) = \sin(xy)$, $\mathsf{t}(x,y) = 5\cos(\pi xy)$.

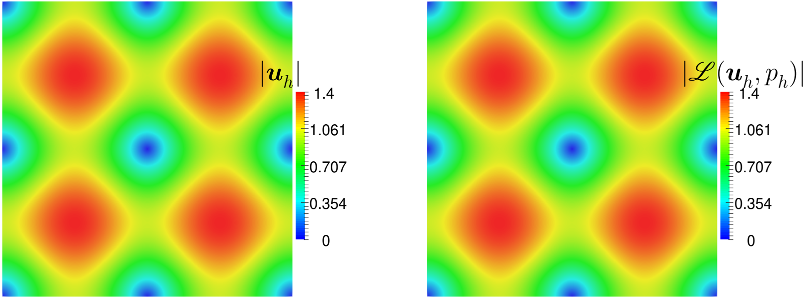

Figure 5: Magnitude of the discrete velocity $\boldsymbol{u}_h$ (left) and post-processed $\mathscr{L}(\boldsymbol{u}_h, p_h)$ (right).

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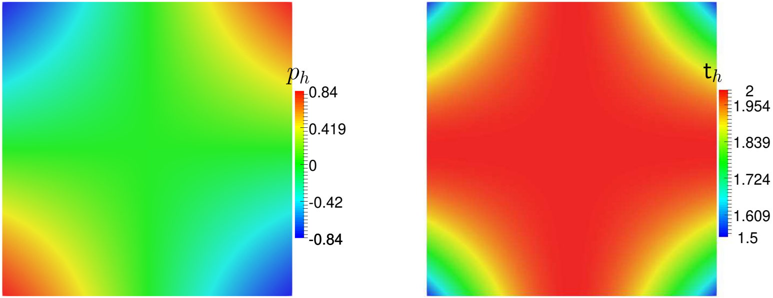

Figure 5: Discrete pressure (left) and temperature $\mathscr{L}(\boldsymbol{u}_h, p_h)$ (right).

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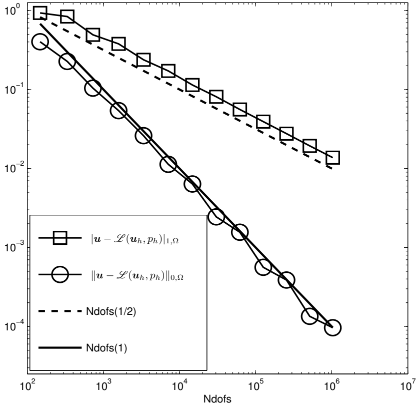
Data: $\varepsilon(\mathsf{t}) = e^{-\mathsf{t}}, \kappa(\mathsf{t}) = e^{\mathsf{t}}$, $\boldsymbol{u}(x,y) = (\sin(\pi y), \cos(\pi x))^T$, $p(x,y) = \sin(xy)$, $\mathsf{t}(x,y) = 5\cos(\pi xy)$.



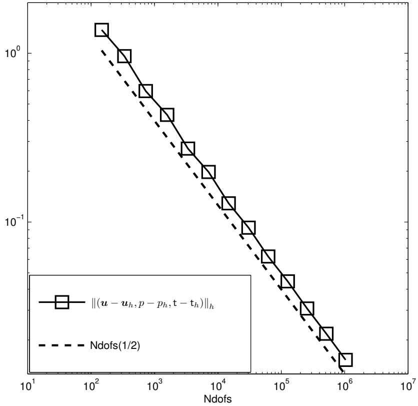


Figure 5: Errors $|\boldsymbol{u} - \mathscr{L}(\boldsymbol{u}_h, p_h)|_{1,h}$ (left) and total error (right).

Numerical results

Data: $\varepsilon(\mathsf{t}) = e^{-\mathsf{t}}, \kappa(\mathsf{t}) = e^{\mathsf{t}}$, $\boldsymbol{u}(x,y) = (\sin(\pi y), \cos(\pi x))^T$, $p(x,y) = \sin(xy)$, $\mathsf{t}(x,y) = 5\cos(\pi xy)$.

Ndof	147	335	727	1583	3367	7119
$\max\limits_{K\in\mathcal{T}_h} \lvert \operatorname{div}\mathscr{L}(\boldsymbol{u}_h, p_h)\rvert_K \rvert$	2.64e-16	1.8e-16	1.25e-16	1.39e-16	1.94e-16	1.11e-16

Ndof	14767	30411	62027	125995	254627	513403
$\max\limits_{K\in\mathcal{T}_h} \lvert \operatorname{div}\mathscr{L}(\boldsymbol{u}_h, p_h)\rvert_K \rvert$	8.15e-17	8.67e-17	3.99e-17	6.25e-17	4.94e-17	3.12e-17

Table 1: Maximun absolutele value of the divergence of $\mathscr{L}(\boldsymbol{u}_h, p_h)$

Numerical results

The need for $\mathcal{S}_\mathsf{t} \neq 0$:

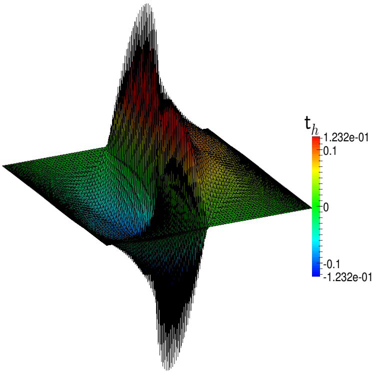


Figure 6: Discrete temperature variable $\mathcal{S}_\mathsf{t} = 0$(left), and CIP stabilization (right), on a mesh with 32768 elements.

Numerical results

The cavity problem :

$$
\begin{aligned}
-Pr\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = (0, Pr Ra \mathsf{t})^T \quad & \text{in } \Omega,\\
\operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega,\\
-\Delta \mathsf{t} + \boldsymbol{u}\cdot\nabla \mathsf{t} = 0 \quad & \text{in } \Omega,
\end{aligned}
$$

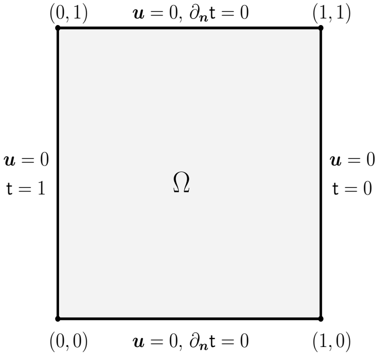


Figure 7: Domain and boundary conditions.

Numerical results

The cavity problem : The stabilisation of the velocity effect

	$\mathcal{S}_u(\cdot,\cdot) = 0$	div-div	CIP
Ndof	$\max_K \lvert \mathbf{div}\, \boldsymbol{u}_h \rvert_K$	$\max_K \lvert \mathbf{div}\, \boldsymbol{u}_h \rvert_K$	$\max_K \lvert \mathbf{div}\, \boldsymbol{u}_h \rvert_K$
35	26.00	23.1	14.2
107	4.65	4.39	3.79
371	0.808	0.749	0.686
691	0.304	0.262	0.244
1379	0.128	0.119	0.114
5315	0.0164	0.0153	0.0151
10435	0.0063	0.00558	0.00562
41347	0.000804	0.000706	0.000712
82692	0.000283	0.000259	0.00026
164611	0.000103	9e-05	9.05e-05
329220	3.59e-05	3.28e-05	3.29e-05
656900	1.3e-05	1.14e-05	1.14e-05

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Numerical results

The cavity problem : The stabilisation of the velocity effect

	$\mathcal{S}_u(\cdot,\cdot) = 0$	div-div	CIP
Ndof	$\max_K \lvert \mathbf{div}\, \mathscr{L}(\boldsymbol{u}_h, p_h)\rvert_K \rvert$	$\max_K \lvert \mathbf{div}\, \mathscr{L}(\boldsymbol{u}_h, p_h)\rvert_K \rvert$	$\max_K \lvert \mathbf{div}\, \mathscr{L}(\boldsymbol{u}_h, p_h)\rvert_K \rvert$
35	1.07e-14	2.49e-14	1.24e-14
107	6.22e-15	5.33e-15	1.07e-14
371	3.89e-15	2.83e-15	3.11e-15
691	1.89e-15	2e-15	2.55e-15
1379	1.72e-15	2.03e-15	1.6e-15
5315	7.22e-16	5.41e-16	6.9e-16
10435	4.58e-16	3.89e-16	7.49e-16
41347	2.5e-16	3.4e-16	6.11e-16
82692	1.96e-16	2.5e-16	3.97e-16
164611	1.21e-16	1.57e-16	4.69e-16
329220	1.35e-16	1.67e-16	3.47e-16
656900	9.71e-17	1.36e-16	3.23e-16

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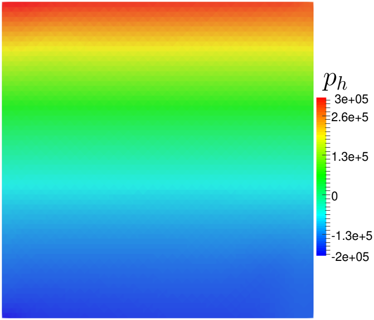
The cavity problem : $Pr = 0.71$, $Ra = 10^6$. The used mesh contains 8,192 elements. We use $S_u(\cdot,\cdot) = 0$.



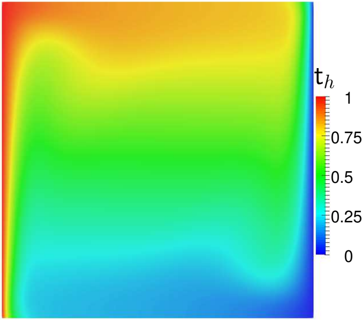


Figure 8: Pressure(left) and temperature (right).

Numerical results

The cavity problem : $Pr = 0.71$, $Ra = 10^6$. The used mesh contains $8,192$ elements. We use $S_u(\cdot,\cdot) = 0$.

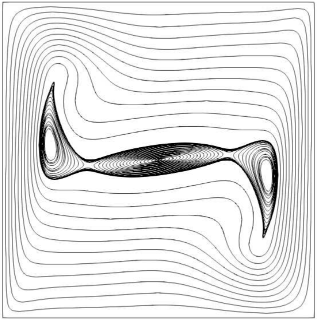


Figure 8: Resolved solution: mesh with 524,000 elements.

Numerical results

The cavity problem : $Pr = 0.71$, $Ra = 10^6$. The used mesh contains $8,192$ elements. We use $S_u(\cdot,\cdot) = 0$.

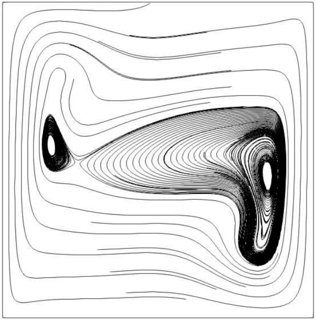
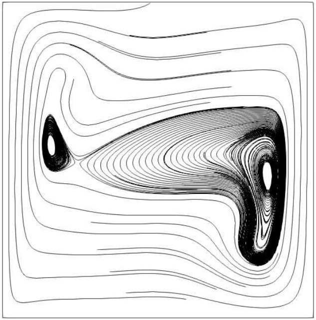


Figure 8: Streamlines of $\boldsymbol{u}_h$ (left) and $\mathscr{L}(\boldsymbol{u}_h, p_h)$ (right).

Numerical results

The cavity problem : $Pr = 0.71, Ra = 10^6$. The used mesh contains 8,192 elements. We use $S_u(\cdot,\cdot) = 0$.



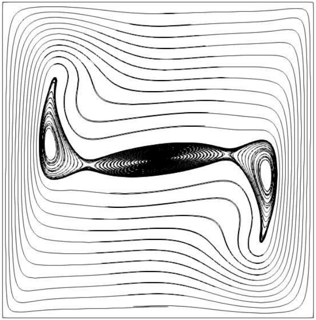


Figure 8: Streamlines of $\boldsymbol{u}_h$ (left) and $\mathscr{L}(\boldsymbol{u}_h, p_h)$ (right).

Numerical results

The cavity problem : $Pr = 0.71, Ra = 10^7$. The used mesh contains $8,192$ elements. We use $S_u(\cdot,\cdot) = 0$.

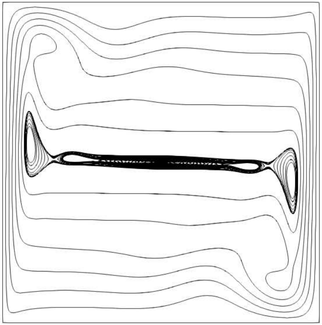


Figure 9: Resolved solution: mesh with 524,000 elements.

The cavity problem : $Pr = 0.71, Ra = 10^7$. The used mesh contains $8,192$ elements. We use $S_u(\cdot,\cdot) = 0$.

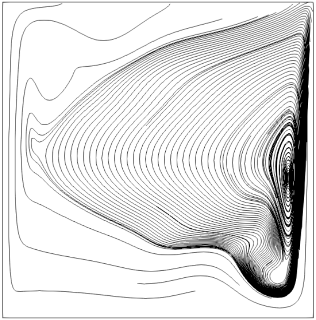


Figure 9: Streamlines of $\boldsymbol{u}_h$ (left) and $\mathscr{L}(\boldsymbol{u}_h, p_h)$ (right).

Numerical results

The cavity problem : $Pr = 0.71, Ra = 10^7$. The used mesh contains $8,192$ elements. We use $S_u(\cdot,\cdot) = 0$.

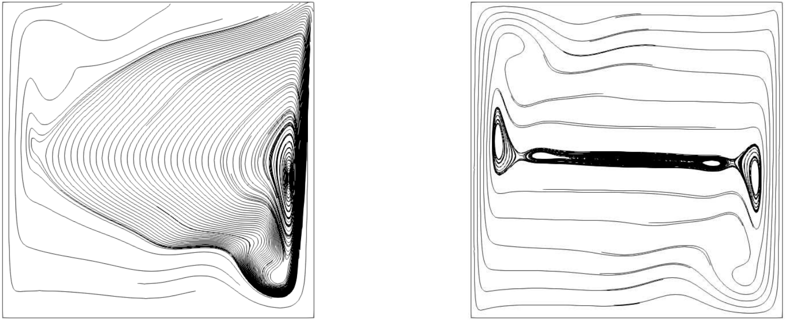


Figure 9: Streamlines of $\boldsymbol{u}_h$ (left) and $\mathscr{L}(\boldsymbol{u}_h, p_h)$ (right).

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Numerical results

The cavity problem : The averaged Nusselt number:

$$\overline{Nu} = \int_\Omega Nu \quad \text{where} \quad Nu := \boldsymbol{u}_x \mathsf{t} - \frac{\partial \mathsf{t}}{\partial x} \,.$$

Ra	$\overline{Nu_c}$	$\overline{Nu_{nc}}$	Chacón et.al.	Vahl-Davis	Massarotti	Manzari
10^3	1.117	1.117	1.118	1.118	1.117	1.074
10^4	2.240	2.241	2.245	2.243	2.243	2.084
10^5	4.499	4.504	4.524	4.519	4.521	4.300
10^6	8.701	8.719	8.852	8.800	8.806	8.743
10^7	16.489	16.491	16.789	–	16.400	13.99

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Conclusions and perspectives

1. A new low-order stabilised finite element method.
2. A hidden div-free velocity field is found and hard-wired into the definition of the method.
3. Viscosity-independent estimates for the Navier-Stokes equation (no stabilisation; no rewriting of the convective term).
4. The hidden solenoidal velocity recovers features of the solution that are not present in the continuous polynomial part.
5. Advantages of a non-conforming method for the price of a conforming one.

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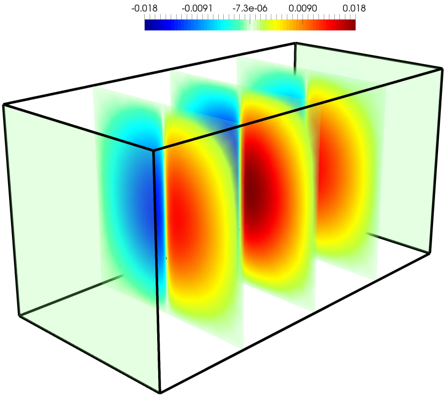
Conclusions and perspectives

Future extensions:

- A posteriori error estimates:

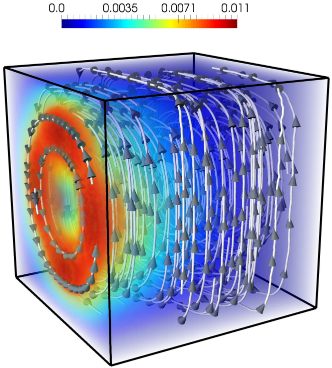


- Non-Newtonian flows.
- Linear Algebra issues.

Conclusions and perspectives

Future extensions:

- A posteriori error estimates:



- Non-Newtonian flows.
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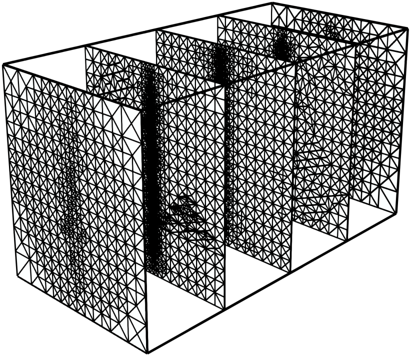
Conclusions and perspectives

Future extensions:

- A posteriori error estimates:



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Conclusions and perspectives

Future extensions:

- A posteriori error estimates:
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