Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Finite Element Methods for Incompressible Flow Problems

Volker John

Curriculum vitae

- 1987 – 1992 study of Mathematics at the Martin–Luther–Universität Halle–Wittenberg
  - Diploma thesis *Numerische Behandlung des Elektrischen Impedanz–Tomographie–Problems*
    advisors: H. Schwetlick, O. Knoth
- 1992 – 1997 scientific assistant at the Institut für Analysis und Numerik of the Otto–von–Guericke–Universität Magdeburg
  - Ph.D. thesis *Parallele Lösung der inkompressiblen Navier–Stokes–Gleichungen auf adaptiv verfeinerten Gittern*
    advisor: L. Tobiska; referees: H.–G. Roos, G. Wittum

Curriculum vitae

- 1997 – 2002 scientific assistant at the Institut für Analysis und Numerik of the Otto–von–Guericke–Universität Magdeburg
  - habilitation thesis *Large Eddy Simulation of Turbulent Incompressible Flows. Analytical and Numerical Results for a Class of LES Models*
    referees: M. Griebel, M.D. Gunzburger, W.J. Layton, L. Tobiska
- 2003 – 2005 scientific assistant at the Institut für Analysis und Numerik of the Otto–von–Guericke–Universität Magdeburg
- 2005 – 2009 Professor for Applied Mathematics, Saarland University, Saarbrücken
- since 01.08.2009 head of the group *Numerical Mathematics and Scientific Computing* at the Weierstrass Institute for Applied Analysis and Stochastics (WIAS) Berlin

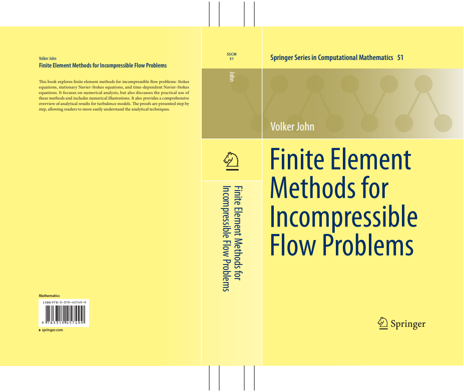
  Professor for Numerical Mathematics, Freie Universität Berlin

WIAS Berlin

- Weierstrass Institute of Applied Analysis and Stochastics
  - goal: project-oriented research in Applied Mathematics
- founded 1992 as successor of the Mathematical Institute of the Academy of Science of the G.D.R.
- member of the Leibniz Association
- $\approx$ 120 researchers in eight research groups
- situated in the center of Berlin
- has been hosting permanent office of the International Mathematical Union (IMU) since 2011

Outline of the Lectures

Literature



- 2016
- xiii+812 pages
- most of the book taught to master students (three semester course)
- extensive review: SIAM Review 60(1), 2018

1. The Navier–Stokes Equations as Model for Incompressible Flows

1 A Model for Incompressible Flows

- conservation laws
  - conservation of linear momentum
  - conservation of mass
- flow variables
  - $\rho(t, \boldsymbol{x})$: density $\left[{}^{\mathrm{kg}}/_{\mathrm{m}^3}\right]$
  - $\boldsymbol{v}(t, \boldsymbol{x})$: velocity $\left[{}^{\mathrm{m}}/_{\mathrm{s}}\right]$
  - $P(t, \boldsymbol{x})$: pressure $\left[{}^{\mathrm{N}}/_{\mathrm{m}^2}\right]$

  assumed to be sufficiently smooth in
- $\Omega \subset \mathbb{R}^3$
- $[0, T]$

- change of fluid in arbitrary volume ω

$$-\frac{\partial}{\partial t}\underbrace{\int_\omega \rho \, d\boldsymbol{x}}_{\text{mass}} = \underbrace{\int_{\partial\omega} \rho \boldsymbol{v} \cdot \boldsymbol{n} \, d\boldsymbol{s}}_{\text{transport through bdry}} = \int_\omega \nabla \cdot (\rho \boldsymbol{v}) \, d\boldsymbol{x}$$

- ω arbitrary $\Longrightarrow$ continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

- incompressibility ($\rho = \text{const}$)

$$\nabla \cdot \boldsymbol{v} = 0$$

- Newton's second law of motion

$$\text{net force} = \text{mass} \ \times \ \text{acceleration}$$

- Newton's second law of motion

$$\text{net force} = \text{mass} \ \times \ \text{acceleration}$$

- conservation of momentum: linear momentum in an arbitrary volume ω is given by

$$\int_\omega \rho \boldsymbol{v}(t, \boldsymbol{x}) \, d\boldsymbol{x} \quad [\mathrm{Ns}]$$

  - formulation analogously to conservation of mass

$$\frac{d}{dt} \int_\omega \rho \boldsymbol{v}(t, \boldsymbol{x}) \, d\boldsymbol{x} = -\int_{\partial\omega} (\rho \boldsymbol{v}) \, (\boldsymbol{v} \cdot \boldsymbol{n}) \, (t, \boldsymbol{s}) \, d\boldsymbol{s} + \int_\omega \boldsymbol{f}_{\text{net}}(t, \boldsymbol{x}) \, d\boldsymbol{x} \; [\mathrm{N}]$$

with

$$\boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{n}) = \begin{pmatrix} v_1 v_1 n_1 + v_1 v_2 n_2 + v_1 v_3 n_3 \\ v_2 v_1 n_1 + v_2 v_2 n_2 + v_2 v_3 n_3 \\ v_3 v_1 n_1 + v_3 v_2 n_2 + v_3 v_3 n_3 \end{pmatrix} = \boldsymbol{v} \boldsymbol{v}^T \boldsymbol{n}$$

- conservation of momentum
  - integration by parts

$$\int_\omega \left(\partial_t\left(\rho\boldsymbol{v}\right) + \nabla\cdot\left(\rho\boldsymbol{v}\boldsymbol{v}^T\right)\right)(t,\boldsymbol{x})\,d\boldsymbol{x} = \int_\omega \boldsymbol{f}_{\mathrm{net}}(t,\boldsymbol{x})\,d\boldsymbol{x}$$

  - product rule

$$\int_\omega \left(\partial_t\rho\boldsymbol{v} + \rho\partial_t\boldsymbol{v} + \boldsymbol{v}\boldsymbol{v}^T\nabla\rho + \rho(\nabla\cdot\boldsymbol{v})\boldsymbol{v} + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right)(t,\boldsymbol{x})\,d\boldsymbol{x}$$
$$= \int_\omega \boldsymbol{f}_{\mathrm{net}}(t,\boldsymbol{x})\,d\boldsymbol{x}$$

  - ρ is constant ($\Longrightarrow\ \nabla\cdot\boldsymbol{v} = 0$)

$$\int_\omega \rho\left(\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right)(t,\boldsymbol{x})\,d\boldsymbol{x} = \int_\omega \boldsymbol{f}_{\mathrm{net}}(t,\boldsymbol{x})\,d\boldsymbol{x}$$

  - ω arbitrary

$$\rho\left(\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = \boldsymbol{f}_{\mathrm{net}} \quad \forall\, t\in(0,T],\ \boldsymbol{x}\in\Omega$$

- acting forces on an arbitrary volume ω:

  sum of external (body) forces

  - gravity

  and internal (molecular) forces

  - pressure
  - viscous drag that a 'fluid element' exerts on the 'adjacent element'
  - contact forces: act only on surface of 'fluid element'

$$\int_\omega \boldsymbol{F}(t, \boldsymbol{x})\, d\boldsymbol{x} + \int_{\partial\omega} \boldsymbol{t}(t, \boldsymbol{s})\, d\boldsymbol{s}$$

$\boldsymbol{t}$ $[\mathrm{N}/\mathrm{m}^2]$ – Cauchy stress vector

- acting forces on an arbitrary volume ω:

  sum of external (body) forces
  - gravity

  and internal (molecular) forces
  - pressure
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  - contact forces: act only on surface of 'fluid element'

$$\int_\omega \boldsymbol{F}(t,\boldsymbol{x})\,d\boldsymbol{x} + \int_{\partial\omega} \boldsymbol{t}(t,\boldsymbol{s})\,d\boldsymbol{s}$$

  $\boldsymbol{t}$ $\left[{}^{\mathrm{N}}/{}_{\mathrm{m}^2}\right]$ – Cauchy stress vector

- principle of Cauchy: internal contact forces depend (geometrically) only on the orientation of the surface

$$\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{n})$$

  $\boldsymbol{n}$ – unit normal vector of the surface pointing outwards of ω

- it can be shown: conservation of linear momentum results in linear dependency on $\boldsymbol{n}$

$$\boldsymbol{t} = \mathbb{S}\boldsymbol{n}$$

$\mathbb{S}(t, \boldsymbol{x})$ $[\mathrm{N}/\mathrm{m}^2]$ – stress tensor, dimension 3×3

- divergence theorem

$$\int_{\partial\omega} \boldsymbol{t}(t, \boldsymbol{s})\, d\boldsymbol{s} = \int_{\omega} \nabla \cdot \mathbb{S}(t, \boldsymbol{x})\, d\boldsymbol{x}$$

- momentum equation

$$\rho \left(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right) = \nabla \cdot \mathbb{S} + \boldsymbol{F} \quad \forall\, t \in (0, T],\ \boldsymbol{x} \in \Omega$$

- model for the stress tensor
  - torque

$$\boldsymbol{M_0} = \int_\omega \boldsymbol{r} \times \boldsymbol{F}\, d\boldsymbol{x} + \int_{\partial\omega} \boldsymbol{r} \times (\mathbb{S}\boldsymbol{n})\ d\boldsymbol{s} \quad [\mathrm{Nm}]$$

    at equilibrium is zero $\Longrightarrow$ symmetry $\mathbb{S} = \mathbb{S}^T$
  - decomposition

$$\mathbb{S} = \mathbb{V} + P\mathbb{I}$$

    $\mathbb{V}$ $\left[{}^{\mathrm{N}}/{}_{\mathrm{m}^2}\right]$ – viscous stress tensor
  - pressure P acts only normal to the surface, directed into ω

$$-\int_{\partial\omega} P\boldsymbol{n}\, d\boldsymbol{s} = -\int_\omega \nabla P\, d\boldsymbol{x} = -\int_\omega \nabla \cdot (P\mathbb{I})\, d\boldsymbol{x}$$

- model for the stress tensor (cont.)
  - viscous stress tensor
    - friction between fluid particles can only occur if the particles move with different velocities
    - $\Longrightarrow$ viscous stress tensor depends on gradient of velocity
    - because of symmetry: on symmetric part of the gradient: velocity deformation tensor

$$\mathbb{D}(\boldsymbol{v}) = \frac{\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T}{2}$$

    - velocity not too large: dependency is linear (Newtonian fluids)

$$\mathbb{V} = 2\mu\mathbb{D}(\boldsymbol{v}) + \left(\zeta - \frac{2\mu}{3}\right)(\nabla \cdot \boldsymbol{v})\,\mathbb{I}$$

μ $\left[{}^{\mathrm{kg}}/_{\mathrm{m\ s}}\right]$ – dynamic or shear viscosity
ζ $\left[{}^{\mathrm{kg}}/_{\mathrm{m\ s}}\right]$ – second order viscosity

- general Navier–Stokes equations

$$
\begin{array}{rcll}
\rho\left(\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) & & & \\
-2\nabla\cdot\left(\mu\mathbb{D}\left(\boldsymbol{v}\right)\right) - \nabla\cdot\left(\left(\zeta - \frac{2\mu}{3}\right)\nabla\cdot\boldsymbol{v}\mathbb{I}\right) + \nabla P & = & \boldsymbol{F} & \text{in } (0,T]\times\Omega, \\
\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) & = & 0 & \text{in } (0,T]\times\Omega
\end{array}
$$

- general Navier–Stokes equations

$$
\begin{array}{rcll}
\rho\left(\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) & & & \\
-2\nabla\cdot\left(\mu\mathbb{D}\left(\boldsymbol{v}\right)\right) - \nabla\cdot\left(\left(\zeta - \frac{2\mu}{3}\right)\nabla\cdot\boldsymbol{v}\mathbb{I}\right) + \nabla P & = & \boldsymbol{F} & \text{in } (0,T]\times\Omega, \\
\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) & = & 0 & \text{in } (0,T]\times\Omega
\end{array}
$$

- incompressible flows: incompressible Navier–Stokes equations

$$
\begin{array}{rcll}
\partial_t \boldsymbol{v} - 2\nu\nabla\cdot\mathbb{D}\left(\boldsymbol{v}\right) + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla\dfrac{P}{\rho_0} & = & \dfrac{\boldsymbol{F}}{\rho_0} & \text{in } (0,T]\times\Omega, \\
\nabla\cdot\boldsymbol{v} & = & 0 & \text{in } (0,T]\times\Omega
\end{array}
$$

- Claude Louis Marie Henri Navier (1785 – 1836)
  George Gabriel Stokes (1819 – 1903)

- dimensionless equations needed for (numerical) analysis and numerical simulations
- reference quantities of flow problem
  - $L\ [\mathrm{m}]$ – a characteristic length scale (diameter of a channel, diameter of a body in the flow)
  - $U\ [\mathrm{m/s}]$ – a characteristic velocity scale (inflow velocity)
  - $T^*\ [\mathrm{s}]$ – a characteristic time scale (period in periodic flows)
- transform of variables

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{L}, \quad \boldsymbol{u} = \frac{\boldsymbol{v}}{U}, \quad t = \frac{t'}{T^*}$$

- rescaling

$$\begin{aligned} \frac{L}{UT^*}\partial_t \boldsymbol{u} - \frac{2\nu}{UL}\nabla\cdot\mathbb{D}\left(\boldsymbol{u}\right) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\frac{P}{\rho_0 U^2} &= \frac{L}{\rho_0 U^2}\boldsymbol{F} && \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } (0,T]\times\Omega \end{aligned}$$

- defining

$$p = \frac{P}{\rho_0 U^2}, \quad \mathsf{Re} = \frac{UL}{\nu}, \quad \mathsf{St} = \frac{L}{UT^*}, \quad \boldsymbol{f} = \frac{L}{\rho_0 U^2}\boldsymbol{F}$$

  p – new pressure

  Re – Reynolds number

  St – Strouhal number

  $\boldsymbol{f}$ – new right-hand side

- result

$$\begin{aligned} \mathsf{St}\partial_t \boldsymbol{u} - \frac{2}{\mathsf{Re}}\nabla\cdot\mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } (0,T]\times\Omega \end{aligned}$$

- generally $T^* = L/U \implies \mathsf{St} = 1$

- dimensionless Navier–Stokes equations
  - conservation of linear momentum
  - conservation of mass

$$
\begin{array}{rcll}
\partial_t \boldsymbol{u} - 2\mathrm{Re}^{-1}\nabla\cdot\mathbb{D}(\boldsymbol{u}) + \nabla\cdot(\boldsymbol{u}\boldsymbol{u}^T) + \nabla p & = & \boldsymbol{f} & \text{in } (0,T]\times\Omega, \\
\nabla\cdot\boldsymbol{u} & = & 0 & \text{in } (0,T]\times\Omega, \\
\boldsymbol{u}(0,\boldsymbol{x}) & = & \boldsymbol{u}_0 & \text{in } \Omega
\end{array}
$$

\+ boundary conditions

- given:
  - $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$: domain
  - T: final time
  - $\boldsymbol{f}$: external forces
  - $\boldsymbol{u}_0$: initial velocity
  - boundary conditions
- parameter: Reynolds number Re

- to compute:
  - velocity $\boldsymbol{u}$, with

$$
\mathbb{D}(\boldsymbol{u}) = \frac{\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T}{2},
$$

velocity deformation tensor

  - pressure p

- Reynolds number

$$\text{Re} = \frac{LU}{\nu} = \frac{\text{convective forces}}{\text{viscous forces}}$$

Osborne Reynolds (1842 – 1912)

- rough classification of flows:
  - Re small: steady-state flow field (if data do not depend on time)
  - Re larger: laminar time-dependent flow field
  - Re very large: turbulent flows

- simplified form (for mathematics)

$$\begin{array}{rcll} \partial_t \boldsymbol{u} - 2\nu \nabla \cdot \mathbb{D}\left(\boldsymbol{u}\right) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p & = & \boldsymbol{f} & \text{in } (0,T] \times \Omega, \\ \nabla \cdot \boldsymbol{u} & = & 0 & \text{in } (0,T] \times \Omega \end{array}$$

$\nu = \mathrm{Re}^{-1}$ – dimensionless viscosity

- simplified form (for mathematics)

$$\begin{aligned} \partial_t \boldsymbol{u} - 2\nu \nabla \cdot \mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } (0,T] \times \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } (0,T] \times \Omega \end{aligned}$$

$\nu = \mathrm{Re}^{-1}$ – dimensionless viscosity

- alternative expression of viscous term (due to $\nabla \cdot \boldsymbol{u} = 0$)

$$2\nabla \cdot \mathbb{D}(\boldsymbol{u}) = \Delta \boldsymbol{u}$$

- alternative expression of convective term (due to $\nabla \cdot \boldsymbol{u} = 0$)

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla \cdot (\boldsymbol{u}\boldsymbol{u}^T)$$

- special cases
  - steady-state Navier–Stokes equations: stationary flow fields

$$
\begin{aligned}
-\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega,\\
\nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega
\end{aligned}
$$

- special cases
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$$\begin{array}{rcll} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p & = & \boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} & = & 0 & \text{in } \Omega \end{array}$$

  - Oseen equations: convection field known (appears in numerical algorithms, for analysis)

$$\begin{array}{rcll} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u} + \nabla p + c\boldsymbol{u} & = & \boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} & = & 0 & \text{in } \Omega \end{array}$$

- special cases
  - steady-state Navier–Stokes equations: stationary flow fields

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  - Oseen equations: convection field known (appears in numerical algorithms, for analysis)

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u} + \nabla p + c\boldsymbol{u} &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

  - Stokes equations: no convection (appears in numerical algorithms)

$$\begin{aligned} -\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

- boundary conditions
  - Dirichlet boundary conditions (inflows)

$$\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{g}(t,\boldsymbol{x}) \text{ in } (0,T] \times \Gamma_{\text{diri}} \subset \Gamma$$

$\boldsymbol{g}(t,\boldsymbol{x}) = \boldsymbol{0}$ – no slip boundary condition (walls)

$$\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{0} \iff \boldsymbol{u}(t,\boldsymbol{x}) \cdot \boldsymbol{n} = 0,\ \boldsymbol{u}(t,\boldsymbol{x}) \cdot \boldsymbol{t}_1 = 0,\ \ \boldsymbol{u}(t,\boldsymbol{x}) \cdot \boldsymbol{t}_2 = 0$$

no penetration, no slip

- boundary conditions
  - Dirichlet boundary conditions (inflows)

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    no penetration, no slip
  - free slip boundary condition (e.g., symmetry planes)

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{n} \quad &= \quad g \quad \text{in } (0,T] \times \Gamma_{\text{slip}} \subset \Gamma, \\ \boldsymbol{n}^T \mathbb{S} \boldsymbol{t}_k \quad &= \quad 0 \quad \text{in } (0,T] \times \Gamma_{\text{slip}}, \quad 1 \le k \le d-1 \end{aligned}$$

    $g = 0$ – no penetration

- boundary conditions (cont.)
  - do-nothing boundary conditions (outflow)

$$\mathbb{S}\boldsymbol{n} = \mathbf{0} \ \text{ in } \ (0,T] \times \Gamma_{\text{outf}} \subset \Gamma$$

- boundary conditions (cont.)
  - do-nothing boundary conditions (outflow)

$$\mathbb{S}\boldsymbol{n} = \boldsymbol{0} \text{ in } (0,T] \times \Gamma_{\text{outf}} \subset \Gamma$$

  - periodic boundary conditions (only for analysis, $\Omega = (0,l)^d$)

$$\boldsymbol{u}(t, \boldsymbol{x} + l\boldsymbol{e}_i) = \boldsymbol{u}(t, \boldsymbol{x}) \quad \forall\, (t, \boldsymbol{x}) \in (0,T] \times \Gamma$$

- difficulties for mathematical analysis and numerical simulations
  - coupling of velocity and pressure
  - nonlinearity of the convective term
  - the convective term dominates the viscous term, i.e., ν is small

2. Finite Element Spaces for Linear Saddle Point Problems

- motivation
  - iterative solution of Navier–Stokes equations leads to linear systems of equations
  - linear systems have special form: saddle point problem (no pressure contribution in second equation)
  - sufficient and necessary condition on unique solvability needed
  - can be derived in abstract form, see [1,2]

[1] Girault, Raviart: *Finite Element Methods for Navier-Stokes Equations* 1986

[2] J.: *Finite Element Methods for Incompressible Flow Problems* 2016, Chapter 3.1

- spaces: V, Q – real Hilbert spaces
- bilinear forms:

$$a(\cdot,\cdot)\ :\ V \times V \to \mathbb{R}, \quad b(\cdot,\cdot)\ :\ V \times Q \to \mathbb{R}$$

- linear problem: Find $(u,p) \in V \times Q$ such that for given $(f,r) \in V' \times Q'$

$$\begin{aligned} a(u,v) + b(v,p) &= \langle f, v\rangle_{V',V} \quad \forall\, v \in V, \\ b(u,q) &= \langle r, q\rangle_{Q',Q} \quad \forall\, q \in Q \end{aligned}$$

- conditions on the spaces and bilinear forms necessary

- associated linear operators

$$
\begin{array}{lll}
A \in \mathcal{L}(V, V') & \text{defined by} & \langle Au, v\rangle_{V',V} = a(u,v) \quad \forall\, u, v \in V \\
B \in \mathcal{L}(V, Q') & \text{defined by} & \langle Bu, q\rangle_{Q',Q} = b(u,q) \quad \forall\, u \in V,\ \forall\, q \in Q
\end{array}
$$

- dual operator: $B' \in \mathcal{L}(Q, V')$ defined by

$$
\langle B'q, v\rangle_{V',V} = \langle Bv, q\rangle_{Q',Q} = b(v,q) \quad \forall\, v \in V,\ \forall\, q \in Q
$$

- linear problem in operator form: Find $(u,p) \in V \times Q$ such that

$$
\begin{array}{llcll}
Au & +B'p & = & f & \text{in } V', \\
Bu & & = & r & \text{in } Q'
\end{array}
$$

- for finite-dimensional spaces, problem can be written in matrix-vector form

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}, \quad \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \in \mathbb{R}^{(n_V+n_Q)\times(n_V+n_Q)},$$

  unique solution $\Longleftrightarrow$ matrix has full rank
- necessary condition: $n_Q \le n_V$
  - last rows of the system matrix span space of at most dimension n_V
- assume that A is non-singular, then the system matrix is non-singular if and only if B has full rank, i.e., $\operatorname{rank}(B) = n_Q$
- $\operatorname{rank}(B) = n_Q$ if and only if

$$\inf_{\underline{q}\in\mathbb{R}^{n_Q}\setminus\underline{0}} \sup_{\underline{v}\in\mathbb{R}^{n_V}\setminus\underline{0}} \frac{\underline{v}^T B^T \underline{q}}{\|\underline{v}\|_2 \, \|\underline{q}\|_2} \ge \beta > 0$$

  - proof much simpler as for infinite-dimensional case, board p. 29

- spaces
  - $V_0 := V(0) = \ker(B), \quad V = V_0^\perp \oplus V_0$
  - $\tilde{V}' = \{\phi \in V' \ : \ \langle \phi, v\rangle_{V',V} = 0 \ \forall\, v \in V_0\} \subset V'$
- inf-sup condition: The three following properties are equivalent:

  i) There exists a constant $\beta_{\text{is}} > 0$ such that

$$\inf_{q\in Q} \sup_{v\in V} \frac{b(v,q)}{\|v\|_V \, \|q\|_Q} \ge \beta_{\text{is}}.$$

  ii) The operator B' is an isomorphism from Q onto $\tilde{V}'$ and

$$\|B'q\|_{V'} \ge \beta_{\text{is}} \, \|q\|_Q \quad \forall\, q \in Q.$$

  iii) The operator B is an isomorphism from $V_0^\perp$ onto Q' and

$$\|Bv\|_{Q'} \ge \beta_{\text{is}} \, \|v\|_V \quad \forall\, v \in V_0^\perp.$$

- derived in [1]
- related condition derived already in [2]: Babuška–Brezzi condition
- sometimes: Ladyzhenskaya–Babuška–Brezzi condition, LBB condition

[1] Brezzi: Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge 8, 129–151, 1974

[2] Babuška: Numer. Math. 16, 322–333, 1971

- sufficient and necessary conditions for unique solution of saddle point problem can be formulated with projection operator, see [1]
- sufficient conditions
  - $a(\cdot,\cdot)$ is V_0-elliptic, i.e., there is a constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha \, \|v\|_V^2 \quad \forall \, v \in V_0$$

  - $b(\cdot,\cdot)$ satisfies inf-sup condition

[1] J.: *Finite Element Methods for Incompressible Flow Problems* 2016, Chapter 3.1

- for simplicity: Dirichlet boundary conditions on whole boundary
- velocity space

$$V = H_0^1(\Omega) = \left\{ \boldsymbol{v} \ : \ \boldsymbol{v} \in H^1(\Omega) \text{ with } \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega \right\}$$

with

$$(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\nabla \boldsymbol{v} \cdot \nabla \boldsymbol{w})(\boldsymbol{x})\, d\boldsymbol{x}, \quad \|\boldsymbol{v}\|_V := \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}$$

dual space: $V' = H^{-1}(\Omega)$

- for simplicity: Dirichlet boundary conditions on whole boundary
- velocity space

$$V = H_0^1(\Omega) = \left\{ \boldsymbol{v} \;:\; \boldsymbol{v} \in H^1(\Omega) \text{ with } \boldsymbol{v} = \mathbf{0} \text{ on } \partial\Omega \right\}$$

  with

$$(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\nabla \boldsymbol{v} \cdot \nabla \boldsymbol{w})(\boldsymbol{x})\, d\boldsymbol{x}, \quad \|\boldsymbol{v}\|_V := \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}$$

  dual space: $V' = H^{-1}(\Omega)$
- pressure space

$$Q = L_0^2(\Omega) = \left\{ q \;:\; q \in L^2(\Omega) \text{ with } \int_\Omega q(\boldsymbol{x})\, d\boldsymbol{x} = 0 \right\}$$

  with

$$(q, r) = \int_\Omega (qr)(\boldsymbol{x})\, d\boldsymbol{x}, \quad \|q\|_Q = \|q\|_{L^2(\Omega)}$$

- dual space: $Q' = Q$

- bilinear form for coupling velocity and pressure

$$b(\boldsymbol{v}, q) = -\int_\Omega q \nabla \cdot \boldsymbol{v} \, d\boldsymbol{x} = -(\nabla \cdot \boldsymbol{v}, q) \quad \boldsymbol{v} \in V, \ q \in Q$$

- bilinear form for coupling velocity and pressure

$$b(\boldsymbol{v}, q) = -\int_\Omega q \nabla \cdot \boldsymbol{v} \, d\boldsymbol{x} = -(\nabla \cdot \boldsymbol{v}, q) \quad \boldsymbol{v} \in V, \; q \in Q$$

- divergence operator

$$\mathsf{div} \; : \; V \to \mathrm{range}(\mathsf{div}), \quad \boldsymbol{v} \mapsto \nabla \cdot \boldsymbol{v}$$

- it can be shown: $\mathrm{range}(\mathsf{div}) = Q'$
- associated linear operator: negative divergence operator

$$B \in \mathcal{L}(V, Q'), \quad B = -\mathsf{div}$$

- dual operator: gradient operator

$$\mathrm{grad} \;:\; Q \to \mathrm{range}(\mathrm{grad}), \quad q \mapsto \nabla q$$

with

$$B' \in \mathcal{L}(Q, V'), \quad B' = \mathrm{grad}$$

- dual operator: gradient operator

$$\text{grad} \;:\; Q \to \mathrm{range}(\text{grad}), \quad q \mapsto \nabla q$$

with

$$B' \in \mathcal{L}(Q, V'), \quad B' = \text{grad}$$

- kernel of B: space of weakly divergence-free functions

$$V_0 = V_{\mathrm{div}} = \{\boldsymbol{v} \in V \;:\; (\nabla \cdot \boldsymbol{v}, q) = 0 \; \forall\, q \in Q\}$$

- estimating divergence by gradient

$$\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \le \sqrt{d}\,\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in H^1(\Omega)$$

  - proof: board, p. 45
  - estimate is sharp

- estimating divergence by gradient

$$\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \le \sqrt{d}\,\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in H^1(\Omega)$$

  - proof: board, p. 45
  - estimate is sharp
- estimating divergence by gradient

$$\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \le \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega)$$

  - proof: based on identity

$$-\Delta \boldsymbol{v} = -\nabla\left(\nabla \cdot \boldsymbol{v}\right) + \mathrm{rot}\ \mathrm{rot}\ \boldsymbol{v}$$

    and integration by parts

- estimating divergence by gradient

$$\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \leq \sqrt{d}\,\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in H^1(\Omega)$$

  - proof: board, p. 45
  - estimate is sharp
- estimating divergence by gradient

$$\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \leq \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega)$$

  - proof: based on identity

$$-\Delta \boldsymbol{v} = -\nabla\left(\nabla \cdot \boldsymbol{v}\right) + \operatorname{rot}\operatorname{rot} \boldsymbol{v}$$

    and integration by parts
- boundedness and continuity of $b(\cdot,\cdot)$

$$|b(\boldsymbol{v}, q)| \leq \|\boldsymbol{v}\|_V \, \|q\|_Q$$

  - proof: board, p. 47

- one can show: div is an isomorphism from $V_{\mathrm{div}}^{\perp}$ onto Q
- corollary: each pressure is the divergence of a velocity field:
  for each $q \in Q$ there is a unique $\boldsymbol{v} \in V_{\mathrm{div}}^{\perp} \subset V$ such that

$$\nabla \cdot \boldsymbol{v} = q \quad \text{and} \quad \|q\|_Q \leq \|\boldsymbol{v}\|_V\,, \quad \|\boldsymbol{v}\|_V \leq C\,\|q\|_Q$$

  with C independent of $\boldsymbol{v}$ and q
  - proof: board, p. 50

- one can show: div is an isomorphism from $V_{\mathrm{div}}^{\perp}$ onto Q
- corollary: each pressure is the divergence of a velocity field:
  for each $q \in Q$ there is a unique $\boldsymbol{v} \in V_{\mathrm{div}}^{\perp} \subset V$ such that

$$\nabla \cdot \boldsymbol{v} = q \quad \text{and} \quad \|q\|_Q \le \|\boldsymbol{v}\|_V \,, \quad \|\boldsymbol{v}\|_V \le C \, \|q\|_Q$$

  with C independent of $\boldsymbol{v}$ and q
  - proof: board, p. 50
- V and Q satisfy the inf-sup condition, i.e., there is a $\beta_{\mathrm{is}} > 0$ such that

$$\inf_{q \in Q} \sup_{\boldsymbol{v} \in V} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_V \, \|q\|_Q} \ge \beta_{\mathrm{is}}$$

  - proof: board, p. 51

- idea: replace infinite-dimensional spaces V and Q by finite-dimensional spaces

- idea: replace infinite-dimensional spaces V and Q by finite-dimensional spaces
- finite element spaces
  - V^h – finite element velocity space
  - Q^h – finite element pressure space
  - V^h/Q^h – pair
- conforming finite element spaces: $V^h \subset V$ and $Q^h \subset Q$

- bilinear form $a^h \;:\; V^h \times V^h \to \mathbb{R}$

$$a^h\left(\boldsymbol{v}^h, \boldsymbol{w}^h\right) := \sum_{K \in \mathcal{T}^h} \left(\nabla \boldsymbol{v}^h, \nabla \boldsymbol{w}^h\right)_K \overset{\text{if conf.}}{=} \left(\nabla \boldsymbol{v}^h, \nabla \boldsymbol{w}^h\right)$$

  - $\mathcal{T}^h$ – triangulation of Ω
  - $K \in \mathcal{T}^h$ – mesh cells
- bilinear form $b^h \;:\; V^h \times Q^h \to \mathbb{R}$

$$b^h\left(\boldsymbol{v}^h, q^h\right) := -\sum_{K \in \mathcal{T}^h} \left(\nabla \cdot \boldsymbol{v}^h, q^h\right)_K \overset{\text{if conf.}}{=} -\left(\nabla \cdot \boldsymbol{v}^h, q^h\right)$$

- norm in velocity finite element space

$$\left\|\boldsymbol{v}^h\right\|_{V^h}^2 = \left(\boldsymbol{v}^h, \boldsymbol{v}^h\right)_{V^h} = \sum_{K \in \mathcal{T}^h} \left(\nabla \boldsymbol{v}^h, \nabla \boldsymbol{v}^h\right)_K \overset{\text{if conf.}}{=} \left(\nabla \boldsymbol{v}^h, \nabla \boldsymbol{v}^h\right)$$

- discrete inf-sup condition needed (finite-dimensional spaces are Hilbert spaces)

$$\inf_{q^h\in Q^h\setminus\{0\}} \sup_{\boldsymbol{v}^h\in V^h\setminus\{\mathbf{0}\}} \frac{b^h\left(\boldsymbol{v}^h, q^h\right)}{\left\|\boldsymbol{v}^h\right\|_{V^h}\left\|q^h\right\|_{L^2(\Omega)}} \geq \beta_{\text{is}}^h > 0$$

  - not inherited from inf-sup condition fulfilled by V and Q
  - discussion: board, p. 53

- space of discretely divergence-free functions

$$V_{\mathrm{div}}^h = \left\{ \boldsymbol{v}^h \in V^h \ : \ b^h\left(\boldsymbol{v}^h, q^h\right) = 0 \ \forall \, q^h \in Q^h \right\}$$

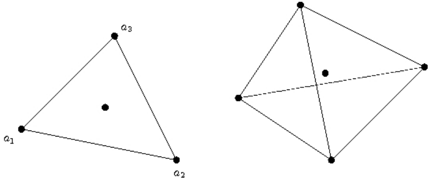
  - generally very hard to construct
- generally $V_{\mathrm{div}}^h \not\subset V_{\mathrm{div}}$
  - finite element velocities not weakly or pointwise divergence-free
  - conservation of mass violated $\Longrightarrow$ Chapter 5
- best approximation estimate for V_{div}^h. Let $\boldsymbol{v} \in V_{\mathrm{div}}$ and let the discrete inf-sup condition hold, then

$$\inf_{\boldsymbol{v}^h \in V_{\mathrm{div}}^h} \left\| \nabla \left(\boldsymbol{v} - \boldsymbol{v}^h\right) \right\|_{L^2(\Omega)} \le \left(1 + \frac{1}{\beta_{\mathrm{is}}^h}\right) \inf_{\boldsymbol{w}^h \in V^h} \left\| \nabla \left(\boldsymbol{v} - \boldsymbol{w}^h\right) \right\|_{L^2(\Omega)}$$

- for certain pairs of finite element spaces estimate with local inf-sup constant [1]

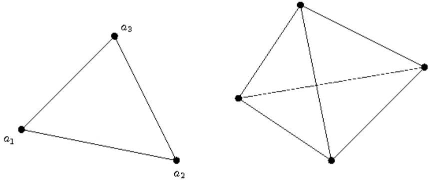
[1] Girault, Scott; Calcolo 40, 1–19, 2003

- piecewise constant finite elements P_0, (Q_0)



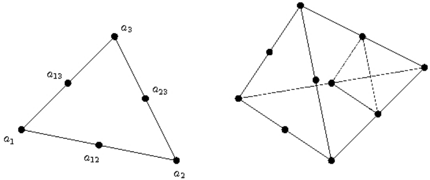
  one degree of freedom (d.o.f.) per mesh cell

- continuous piecewise linear finite elements P_1



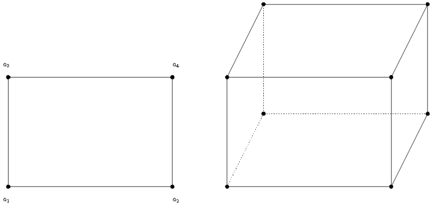
  d d.o.f. per mesh cell

- continuous piecewise quadratic finite elements P_2



$(d+1)(d+2)/2$ d.o.f. per mesh cell

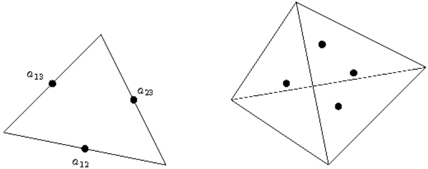
- continuous piecewise bilinear finite elements Q_1



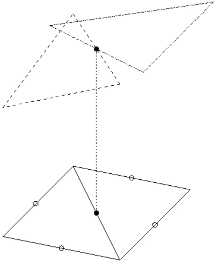
2^d d.o.f. per mesh cell

- and so on for continuous finite elements of higher order

- nonconforming linear finite elements P_1^{nc}, Crouzeix, Raviart (1973)

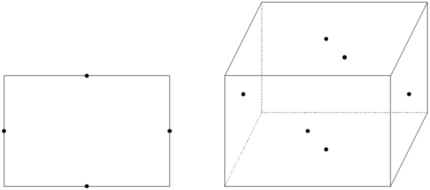


continuous only in barycenters of faces



$d+1$ d.o.f. per mesh cell

- rotated bilinear finite element Q_1^{rot}, Rannacher, Turek (1992)



  - continuous only in barycenters of faces
  - $2d$ d.o.f. per mesh cell
- discontinuous linear finite element P_1^{disc}
  - defined by integral nodal functionals
    e.g., $\varphi^h \in P_1^{\text{disc}}$ if φ^h is linear on a mesh cell K (2d) and

$$\int_K \varphi^h(\boldsymbol{x})\, d\boldsymbol{x} = 0, \int_K x\varphi^h(\boldsymbol{x})\, d\boldsymbol{x} = 1, \int_K y\varphi^h(\boldsymbol{x})\, d\boldsymbol{x} = 0$$

  - $d+1$ d.o.f. per mesh cell

- criterion for violation of discrete inf-sup condition: there is non-trivial $q^h \in Q^h$ such that

$$b^h\left(\boldsymbol{v}^h, q^h\right) = 0 \quad \forall\, \boldsymbol{v}^h \in V^h$$

$\Longrightarrow$

$$\sup_{\boldsymbol{v}^h \in V^h \setminus \{\mathbf{0}\}} \frac{b^h\left(\boldsymbol{v}^h, q^h\right)}{\left\|\boldsymbol{v}^h\right\|_{V^h}} = 0$$

- criterion for violation of discrete inf-sup condition: there is non-trivial $q^h \in Q^h$ such that

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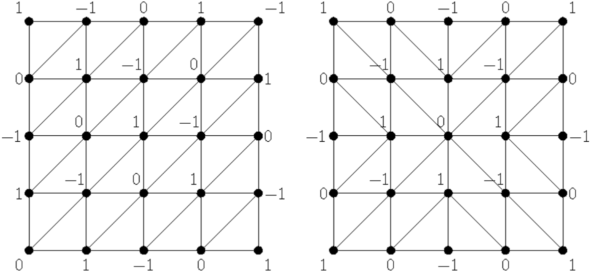
$\Longrightarrow$

$$\sup_{\boldsymbol{v}^h \in V^h \setminus \{\mathbf{0}\}} \frac{b^h\left(\boldsymbol{v}^h, q^h\right)}{\left\|\boldsymbol{v}^h\right\|_{V^h}} = 0$$

- P_1/P_1 pair of finite element spaces violates discrete inf-sup condition
  - counter example: checkerboard instability, board p. 63



- other pairs which violated discrete inf-sup condition
  - P_1/P_0
  - Q_1/Q_0
  - P_k/P_k, $k \geq 1$
  - Q_k/Q_k, $k \geq 1$
  - $P_k/P_{k-1}^{\mathrm{disc}}$, $k \geq 2$, on a special macro cell
- summary:
  - many easy to implement pairs violate discrete inf-sup condition
  - different finite element spaces for velocity and pressure necessary

- pairs which fulfill discrete inf-sup condition
  - $P_k/P_{k-1}, Q_k/Q_{k-1}$: Taylor–Hood finite elements [1]
    - proofs: 2D, $k = 2$ [2], general [3,4]
  - $Q_k/P_{k-1}^{\rm disc}$
  - $P_k/P_{k-1}^{\rm disc}, k \ge d$, on very special meshes (Scott–Vogelius element)
  - $P_1^{\rm bubble}/P_1$, MINI element
  - $P_k^{\rm bubble}/P_{k-1}^{\rm disc}$ [5]
  - $P_1^{\rm nc}/P_0$, Crouzeix–Raviart element [6]
  - $Q_1^{\rm rot}/Q_0$, Rannacher–Turek element [7]
  - $\vdots$

[1] Taylor, Hood; Comput. Fluids 1, 73-100, 1973

[2] Verfürth; RAIRO Anal. Numér. 18, 175–182, 1984

[3] Boffi; Math. Models Methods Appl. Sci. 4,223–235, 1994

[4] Boffi; SIAM J. Numer. Anal. 34, 664–670, 1997

[5] Bernardi, Raugel; Math. Comp. 44, 71–79, 1985

[6] Crouzeix, Raviart; RAIRO. Anal. Numér. 7, 33–76, 1973

[7] Rannacher, Turek; Numer. Meth. Part. Diff. Equ. 8, 97–111, 1992

- techniques for proving the discrete inf-sup condition
  - construction of Fortin operator [1]
  - using projection to piecewise constant pressure [2]
  - macroelement techniques [3,4]
  - criterion from [5] for continuous finite element pressure: to show

$$\sup_{\boldsymbol{v}^h \in V^h \setminus \{\mathbf{0}\}} \frac{b(\boldsymbol{v}^h, q^h)}{\|\boldsymbol{v}^h\|_V} \ge \beta_2 \left(\sum_{K \in \mathcal{T}^h} h_K^2 \left\|\nabla q^h\right\|_{L^2(K)}^2 \right)^{1/2} \quad \forall\, q^h \in Q^h$$

  - survey in [6]

[1] Fortin; RAIRO Anal. Numér. 11, 341–354, 1977

[2] Brezzi, Bathe; Comput. Methods Appl. Mech. Engrg. 82, 27–57, 1990

[3] Boland, Nicolaides; SIAM J. Numer. Anal. 20, 722–731, 1983

[4] Stenberg; Math. Comp. 32, 9–23, 1984

[5] Verfürth; RAIRO Anal. Numér. 18, 175–18, 1984

[6] Boffi, Brezzi, Fortin; Lecture Notes in Mathematics 1939, Springer, 45–100, 2008

- connection between the continuous and the discrete inf-sup condition
  - let V, Q satisfy the continuous inf-sup condition
  - let $V^h \subset V$ and $Q^h \subset Q$
  - $\Longrightarrow V^h$ and Q^h satisfy the discrete inf-sup condition if and only if there exists a constant $\gamma^h > 0$, independent of h, such that for all $\boldsymbol{v} \in V$ there is an element $P_{\text{For}}^h \boldsymbol{v} \in V^h$ with

$$b\left(\boldsymbol{v}, q^h\right) = b\left(P_{\text{For}}^h \boldsymbol{v}, q^h\right) \quad \forall\, q^h \in Q^h \quad \text{and} \quad \left\|P_{\text{For}}^h \boldsymbol{v}\right\|_V \leq \gamma^h \left\|\boldsymbol{v}\right\|_V$$

  - proof $\Longleftarrow$: board p. 73

- general approach for constructing Fortin operator

$$P_{\mathrm{For}}^h \in \mathcal{L}(V, V^h) \quad \boldsymbol{v} \mapsto P_{\mathrm{Cle}}^h \boldsymbol{v} + P_2^h \left(\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v} \right)$$

  with
  - P_{Cle}^h – Clément operator (modification which preserves homogeneous Dirichlet boundary conditions)
  -

$$\left\| P_2^h \boldsymbol{v} \right\|_{H^1(K)} \le C \left(h_K^{-1} \left\| \boldsymbol{v} \right\|_{L^2(K)} + \left| \boldsymbol{v} \right|_{H^1(K)} \right), \quad \forall\, K \in \mathcal{T}^h, \forall\, \boldsymbol{v} \in V$$

- used to prove, e.g., inf-sup condition for MINI element $P_1 \oplus V_{\mathrm{bub}}^h / P_1$ [1],
- Fortin operator approach can be extended to Crouzeix–Raviart element P_1^{nc} / P_0 and Rannacher–Turek element Q_1^{rot} / Q_0

[1] Arnold, Brezzi, Fortin; Calcolo 21, 337–344, 1984

- goal: reduce proof of global inf-sup condition to proof of local inf-sup conditions
- Boland, Nicolaides (1983)
  - $Q_k/P_{k-1}^{\text{disc}}$, $k \geq 2$ (unmapped [1], mapped [2])
- Stenberg (1984)
  - Taylor–Hood P_k/P_{k-1}, Q_k/Q_{k-1}, $k \geq 2$ [3]
  - unmapped Q_2/P_1^{disc} in 2d [4]
  - Scott–Vogelius $P_k/P_{k-1}^{\text{disc}}$, $k \geq 3$, in 3d on barycentric meshes [5]
- general theory for both approaches somewhat technical

[1] Girault, Raviart; Springer-Verlag, 1986

[2] Matthies, Tobiska; Computing 69, 119–139, 2002

[3] Stenberg; Math. Comp. 54, 495–508, 1990

[4] Stenberg; Math. Comp. 32, 9–23, 1984

[5] Zhang; Math. Comp. 74, 543–554, 2005

- from Riesz representation theorem: there is $\boldsymbol{v}_b^h \in V^h$ such that for fixed $q^h \in Q^h$

$$\left(\boldsymbol{v}_b^h, \boldsymbol{v}^h\right)_{V^h} = b^h\left(\boldsymbol{v}^h, q^h\right) \quad \forall\, \boldsymbol{v}^h \in V^h$$

- it follows for all $\boldsymbol{v}^h \in V^h$

$$b^h\left(\boldsymbol{v}^h, q^h\right) \le \left\|\boldsymbol{v}_b^h\right\|_{V^h} \left\|\boldsymbol{v}^h\right\|_{V^h} \quad \Longrightarrow \quad \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b^h\left(\boldsymbol{v}^h, q^h\right)}{\left\|\boldsymbol{v}^h\right\|_{V^h}} \le \left\|\boldsymbol{v}_b^h\right\|_{V^h}$$

- supremum is attained, since

$$\frac{b^h\left(\boldsymbol{v}_b^h, q^h\right)}{\left\|\boldsymbol{v}_b^h\right\|_{V^h}} = \left\|\boldsymbol{v}_b^h\right\|_{V^h} \quad \Longrightarrow \quad \boldsymbol{v}_b^h = \operatorname*{arg\,sup}_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b^h\left(\boldsymbol{v}^h, q^h\right)}{\left\|\boldsymbol{v}^h\right\|_{V^h}}$$

- it follows

$$\left(\beta_{\text{is}}^h\right)^2 = \inf_{q^h \in Q^h \setminus \{0\}} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{\left(b^h\left(\boldsymbol{v}^h, q^h\right)\right)^2}{\left\|\boldsymbol{v}^h\right\|_{V^h}^2 \left\|q^h\right\|_Q^2} = \inf_{q^h \in Q^h \setminus \{0\}} \frac{\left\|\boldsymbol{v}_b^h\right\|_{V^h}^2}{\left\|q^h\right\|_Q^2}$$

- equip spaces with bases $\Longrightarrow$ Gramian matrices

$$M_V = \left(\left(\boldsymbol{\phi}_j^h, \boldsymbol{\phi}_i^h\right)_{V^h}\right)_{i,j=1}^{N_u}, \quad M_Q = \left(\left(\psi_j^h, \psi_i^h\right)_{Q_h}\right)_{i,j=1}^{N_p}$$

and bilinear forms and norms

$$\left(\boldsymbol{v}_b^h, \boldsymbol{v}^h\right)_{V^h} = \underline{b}^T M_V^T \underline{v}, \quad b^h\left(\boldsymbol{v}^h, q^h\right) = \underline{v}^T B^T \underline{q}, \quad \left\|\boldsymbol{v}_b^h\right\|_{V^h}^2 = \underline{b}^T M_V^T \underline{b}$$

- one obtains from Riesz condition

$$\underline{b}^T M_V^T \underline{v} = \underline{v}^T B^T \underline{q} \quad \Longleftrightarrow \quad \underline{v}^T M_V \underline{b} = \underline{v}^T B^T \underline{q} \quad \forall\, \underline{v} \in \mathbb{R}^{N_u},$$

from what follows that

$$M_V \underline{b} = B^T \underline{q} \quad \Longrightarrow \quad \underline{b} = M_V^{-1} B^T \underline{q}$$

- inserting in discrete inf-sup condition

$$\left(\beta_{\text{is}}^h\right)^2 = \inf_{\underline{q}\in\mathbb{R}^{N_q}\setminus\{\underline{0}\}} \frac{\underline{b}^T M_V^T \underline{b}}{\underline{q}^T M_Q^T \underline{q}} = \inf_{\underline{q}\in\mathbb{R}^{N_q}\setminus\{\underline{0}\}} \frac{\underline{q}^T B M_V^{-T} M_V^T M_V^{-1} B^T \underline{q}}{\underline{q}^T M_Q^T \underline{q}}$$

$$= \inf_{\underline{q}\in\mathbb{R}^{N_q}\setminus\{\underline{0}\}} \frac{\underline{q}^T B M_V^{-1} B^T \underline{q}}{\underline{q}^T M_Q^T \underline{q}}$$

- M_V and M_Q are symmetric and positive definite

$$\frac{\left(\underline{q}^T M_Q^{1/2}\right)\left(M_Q^{-1/2} B M_V^{-T/2}\right)\left(M_V^{-1/2} B^T M_Q^{-T/2}\right)\left(M_Q^{T/2}\underline{q}\right)}{\left(\underline{q}^T M_Q^{1/2}\right)\left(M_Q^{T/2}\underline{q}\right)}$$

$\Longrightarrow$ Rayleigh quotient

- infimum is smallest eigenvalue of

$$\left(M_Q^{-1/2} B M_V^{-T/2}\right)\left(M_V^{-1/2} B^T M_Q^{-T/2}\right)\left(M_Q^{T/2}\underline{q}\right) = \lambda\left(M_Q^{T/2}\underline{q}\right)$$

or (multiply with $M_Q^{1/2}$)

$$B M_V^{-1} B^T \underline{q} = \lambda M_Q \underline{q}$$

- discrete inf-sup constant is square root of smallest eigenvalue of this generalized eigenvalue problem

3. Finite Element Error Analysis of the Stokes Equations

- continuous equation

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega \end{aligned} \tag{1}$$

  for simplicity: homogeneous Dirichlet boundary conditions
- difficulty: coupling of velocity and pressure
- properties
  - linear
  - form

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

    becomes (1) by rescaling with new pressure, right-hand side

- finite element problem: Find $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ such that

$$\begin{array}{rcll} a^h\left(\boldsymbol{u}^h, \boldsymbol{v}^h\right) + b^h\left(\boldsymbol{v}^h, p^h\right) & = & \left(\boldsymbol{f}, \boldsymbol{v}^h\right) & \forall\, \boldsymbol{v}^h \in V^h, \\ b^h\left(\boldsymbol{u}^h, q^h\right) & = & 0 & \forall\, q^h \in Q^h \end{array}$$

with

$$a^h\left(\boldsymbol{v}^h, \boldsymbol{w}^h\right) = \sum_{K \in \mathcal{T}^h} \left(\nabla \boldsymbol{v}^h, \nabla \boldsymbol{w}^h\right)_K, \quad b^h\left(\boldsymbol{v}^h, q^h\right) = -\sum_{K \in \mathcal{T}^h} \left(\nabla \cdot \boldsymbol{v}^h, q^h\right)_K$$

- finite element problem: Find $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ such that

$$\begin{array}{rcll} a^h\left(\boldsymbol{u}^h, \boldsymbol{v}^h\right) + b^h\left(\boldsymbol{v}^h, p^h\right) & = & \left(\boldsymbol{f}, \boldsymbol{v}^h\right) & \forall\, \boldsymbol{v}^h \in V^h, \\ b^h\left(\boldsymbol{u}^h, q^h\right) & = & 0 & \forall\, q^h \in Q^h \end{array}$$

  with

$$a^h\left(\boldsymbol{v}^h, \boldsymbol{w}^h\right) = \sum_{K\in\mathcal{T}^h} \left(\nabla \boldsymbol{v}^h, \nabla \boldsymbol{w}^h\right)_K, \quad b^h\left(\boldsymbol{v}^h, q^h\right) = -\sum_{K\in\mathcal{T}^h} \left(\nabla\cdot \boldsymbol{v}^h, q^h\right)_K$$

- only conforming inf-sup stable finite element spaces
  - $V^h \subset V$ and $Q^h \subset Q$
  - $$\inf_{q^h\in Q^h\setminus\{0\}} \sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}} \frac{b^h\left(\boldsymbol{v}^h, q^h\right)}{\|\boldsymbol{v}^h\|_{V^h}\, \|q^h\|_{L^2(\Omega)}} \ge \beta_{\mathrm{is}}^h > 0$$

- existence and uniqueness of a solution
  - apply theory of linear saddle point problems

- existence and uniqueness of a solution
  - apply theory of linear saddle point problems
- stability

$$\left\|\nabla \boldsymbol{u}^h\right\|_{L^2(\Omega)} \le \|\boldsymbol{f}\|_{H^{-1}(\Omega)}, \quad \left\|p^h\right\|_{L^2(\Omega)} \le \frac{2}{\beta_{\mathrm{is}}^h} \|\boldsymbol{f}\|_{H^{-1}(\Omega)}$$

  - proof: same as for continuous problem, board, p. 139

- existence and uniqueness of a solution
  - apply theory of linear saddle point problems
- stability

$$\left\|\nabla \boldsymbol{u}^h\right\|_{L^2(\Omega)} \le \|\boldsymbol{f}\|_{H^{-1}(\Omega)}, \quad \left\|p^h\right\|_{L^2(\Omega)} \le \frac{2}{\beta_{\text{is}}^h}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}$$

  - proof: same as for continuous problem, board, p. 139
- goal of finite element error analysis: estimate error by best approximation errors
  - best approximation errors depend only on finite element spaces, not on problem
  - estimates for best approximation errors are known (interpolation errors)

- existence and uniqueness of a solution
  - apply theory of linear saddle point problems
- stability

$$\left\|\nabla \boldsymbol{u}^h\right\|_{L^2(\Omega)} \le \|\boldsymbol{f}\|_{H^{-1}(\Omega)}, \quad \left\|p^h\right\|_{L^2(\Omega)} \le \frac{2}{\beta_{\mathrm{is}}^h} \|\boldsymbol{f}\|_{H^{-1}(\Omega)}$$

  - proof: same as for continuous problem, board, p. 139
- goal of finite element error analysis: estimate error by best approximation errors
  - best approximation errors depend only on finite element spaces, not on problem
  - estimates for best approximation errors are known (interpolation errors)
- reduction to a problem on the space of discretely divergence-free functions

$$a\left(\boldsymbol{u}^h, \boldsymbol{v}^h\right) = \left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) = \left(\boldsymbol{f}, \boldsymbol{v}^h\right) \ \forall\ \boldsymbol{v}^h \in V_{\mathrm{div}}^h$$

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$, bounded, polyhedral, Lipschitz-continuous boundary
  - general case: $V_{\text{div}}^h \not\subset V_{\text{div}}$

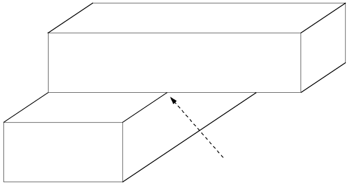
$$\left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)} \le 2 \inf_{\boldsymbol{v}^h \in V_{\text{div}}^h} \left\|\nabla(\boldsymbol{u} - \boldsymbol{v}^h)\right\|_{L^2(\Omega)} + \inf_{q^h \in Q^h} \left\|p - q^h\right\|_{L^2(\Omega)}$$

  - proof: board, p. 146/147
  - velocity error (bound) depends on pressure

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$, bounded, polyhedral, Lipschitz-continuous boundary
  - general case: $V_{\mathrm{div}}^h \not\subset V_{\mathrm{div}}$

$$\begin{aligned}\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)} \quad \leq \quad & 2 \inf_{\boldsymbol{v}^h \in V_{\mathrm{div}}^h} \left\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\right\|_{L^2(\Omega)} \\ & + \inf_{q^h \in Q^h} \left\|p-q^h\right\|_{L^2(\Omega)}\end{aligned}$$

  - proof: board, p. 146/147
  - velocity error (bound) depends on pressure
- polyhedral domain in three dimensions which is not Lipschitz-continuous



- finite element error estimate for the $L^2(\Omega)$ norm of the pressure
  - same assumptions as for previous estimate

$$
\begin{aligned}
\left\|p - p^h\right\|_{L^2(\Omega)} \quad \le \quad & \frac{2}{\beta_{\mathrm{is}}^h} \inf_{\boldsymbol{v}^h \in V_{\mathrm{div}}^h} \left\|\nabla(\boldsymbol{u} - \boldsymbol{v}^h)\right\|_{L^2(\Omega)} \\
& + \left(1 + \frac{2}{\beta_{\mathrm{is}}^h}\right) \inf_{q^h \in Q^h} \left\|p - q^h\right\|_{L^2(\Omega)}
\end{aligned}
$$

  - proof: board, p. 149

- error of the velocity in the $L^2(\Omega)$ norm
  - by Poincaré inequality not optimal

$$\left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \le C \left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)}$$

- error of the velocity in the $L^2(\Omega)$ norm
  - by Poincaré inequality not optimal

$$\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)} \le C\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)}$$

- regular dual Stokes problem: For given $\hat{\boldsymbol{f}} \in L^2(\Omega)$, find $(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}, \xi_{\hat{\boldsymbol{f}}}) \in V \times Q$ such that

$$\begin{aligned} -\Delta\boldsymbol{\phi}_{\hat{\boldsymbol{f}}} + \nabla\xi_{\hat{\boldsymbol{f}}} &= \hat{\boldsymbol{f}} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{\phi}_{\hat{\boldsymbol{f}}} &= 0 \quad \text{in } \Omega \end{aligned}$$

  - regular if mapping

$$\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}, \xi_{\hat{\boldsymbol{f}}}\right) \mapsto -\Delta\boldsymbol{\phi}_{\hat{\boldsymbol{f}}} + \nabla\xi_{\hat{\boldsymbol{f}}}$$

    is an isomorphism from $\left(H^2(\Omega)\cap V\right) \times \left(H^1(\Omega)\cap Q\right)$ onto $L^2(\Omega)$
  - Γ of class C^2
  - bounded, convex polygons in two dimensions

- finite element error estimate for the $L^2(\Omega)$ norm of the velocity
  - same assumptions as for previous estimates
  - dual Stokes problem regular with solution $(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}, \xi_{\hat{\boldsymbol{f}}})$

$$
\begin{aligned}
&\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)} \\
&\quad \leq \left(\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} + \inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\right) \\
&\qquad \times \sup_{\hat{\boldsymbol{f}}\in L^2(\Omega)} \frac{1}{\left\|\hat{\boldsymbol{f}}\right\|_{L^2(\Omega)}} \left[\left(1+\frac{1}{\beta_{\mathrm{is}}^h}\right)\inf_{\boldsymbol{\phi}^h\in V^h}\left\|\nabla\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}-\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}\right. \\
&\qquad\qquad \left. + \inf_{r^h\in Q^h}\left\|\xi_{\hat{\boldsymbol{f}}}-r^h\right\|_{L^2(\Omega)}\right]
\end{aligned}
$$

  - velocity error (bound) depends on pressure

- finite element error estimates for conforming pairs of finite element spaces
  - same assumptions on domain as for previous estimates
  - solution sufficiently regular
  - h – mesh width of triangulation
  - spaces
    - $P_k^{\mathrm{bubble}}/P_k$, $k = 1$ (MINI element),
    - P_k/P_{k-1}, Q_k/Q_{k-1}, $k \geq 2$ (Taylor–Hood element),
    - $P_k^{\mathrm{bubble}}/P_{k-1}^{\mathrm{disc}}$, $Q_k/P_{k-1}^{\mathrm{disc}}$, $k \geq 2$

$$\begin{aligned} \left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)} &\leq Ch^k\left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}+\|p\|_{H^k(\Omega)}\right) \\ \left\|p-p^h\right\|_{L^2(\Omega)} &\leq Ch^k\left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}+\|p\|_{H^k(\Omega)}\right) \end{aligned}$$

  - velocity error (bound) depends on pressure

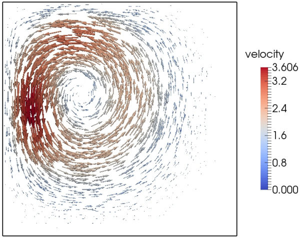
- finite element error estimates for conforming pairs of finite element spaces (cont.)
  - in addition: if dual Stokes problem regular

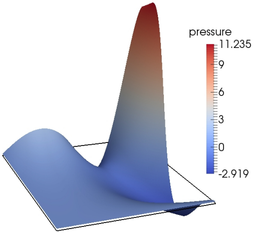
$$\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)} \le Ch^{k+1}\left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}+\|p\|_{H^k(\Omega)}\right)$$

  - velocity error (bound) depends on pressure

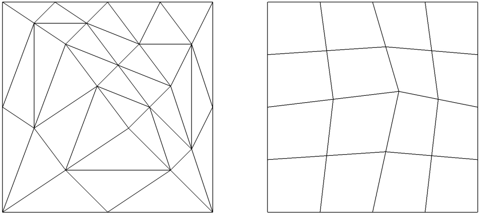
- analytical example which supports the error estimates
- prescribed solution

$$
\begin{aligned}
\boldsymbol{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix} = 200 \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix} \\
p &= 10\left(\left(x-\frac{1}{2}\right)^3 y^2 + (1-x)^3\left(y-\frac{1}{2}\right)^3\right)
\end{aligned}
$$



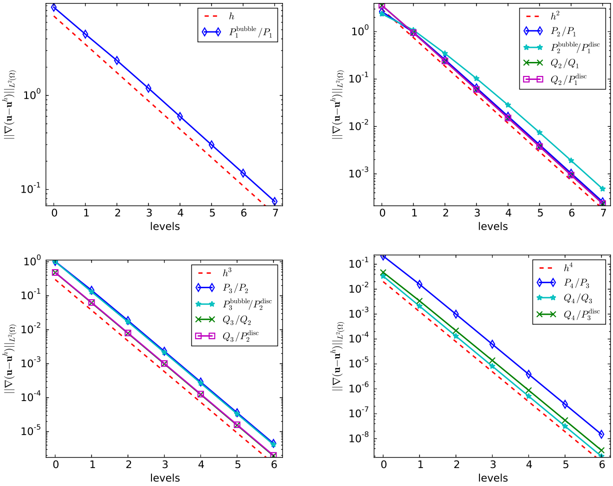


- initial grids (level 0)



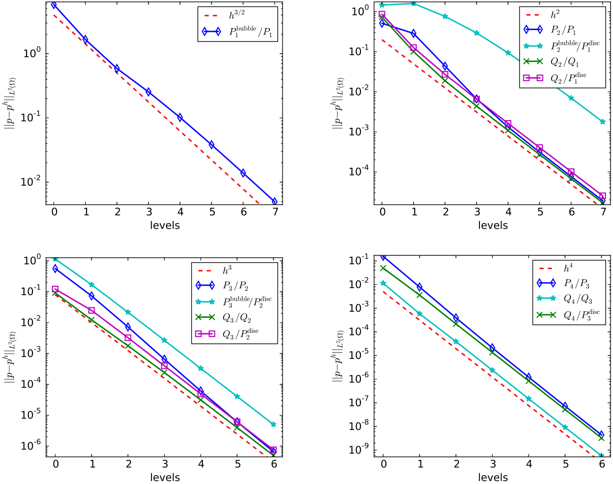
- red refinement

- convergence of the errors $\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)}$ for different discretizations with different orders k

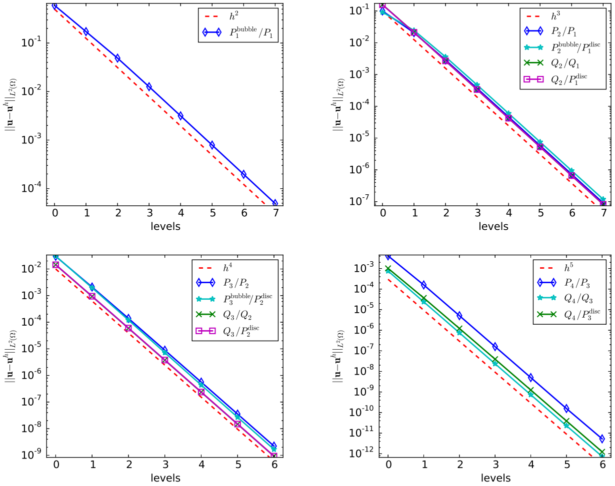


3 Numerical Example Supporting Order of Convergence

- convergence of the errors $\left\|p - p^h\right\|_{L^2(\Omega)}$ for different discretizations with different orders k

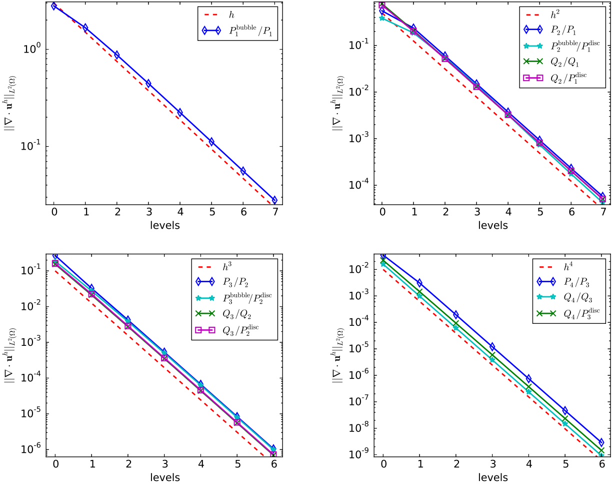


- convergence of the errors $\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)}$ for different discretizations with different orders k



3 Numerical Example Supporting Order of Convergence

- convergence of the errors $\left\|\nabla \cdot \boldsymbol{u}^h\right\|_{L^2(\Omega)}$ for different discretizations with different orders k



- scaled Stokes equations

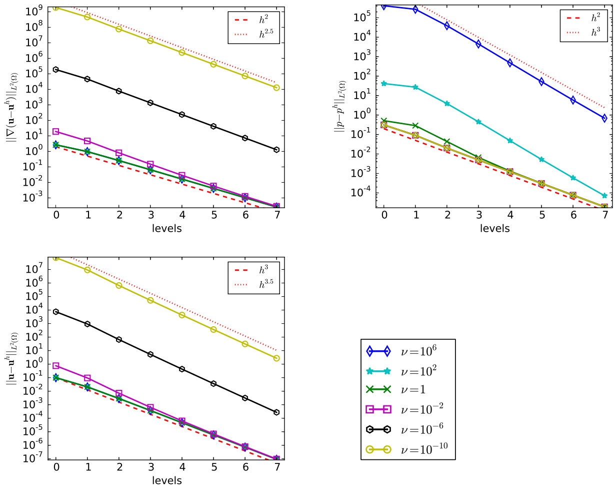
$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \Longleftrightarrow \quad -\Delta\boldsymbol{u} + \nabla\left(\frac{p}{\nu}\right) = \frac{\boldsymbol{f}}{\nu}$$

- error estimates

$$\begin{aligned}
\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)} &\le Ch^{k}\left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \frac{1}{\nu}\|p\|_{H^k(\Omega)}\right)\\
\left\|p-p^h\right\|_{L^2(\Omega)} &\le Ch^{k}\left(\nu\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)}\right)\\
\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)} &\le Ch^{k+1}\left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \frac{1}{\nu}\|p\|_{H^k(\Omega)}\right)
\end{aligned}$$

  - velocity errors (bounds) depend on pressure and inverse of viscosity

- Taylor–Hood pair of finite element spaces P_2/P_1



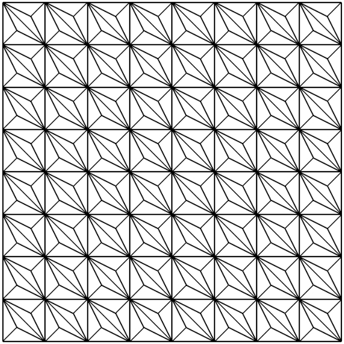
  - velocity errors depend on inverse of viscosity (on coarse grids)

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$, bounded, polyhedral, Lipschitz-continuous boundary
  - special case: $V_{\text{div}}^h \subset V_{\text{div}}$

$$\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)} \le 2 \inf_{\boldsymbol{v}^h \in V_{\text{div}}^h} \left\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\right\|_{L^2(\Omega)}$$

  - proof: board, p. 146, p. 161
  - velocity error does not depend on pressure
- same property for $\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)}$

- most important example: Scott–Vogelius [1] finite element barycentric-refined grid: $P_k/P_{k-1}^{\mathrm{disc}}, k \geq d$

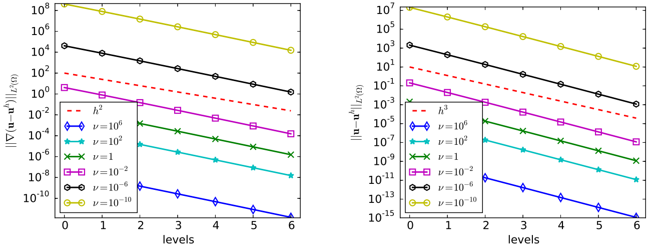


[1] Scott, Vogelius; in Large-scale computations in fluid mechanics, Part 2, 221–244, 1985

- prescribed solution

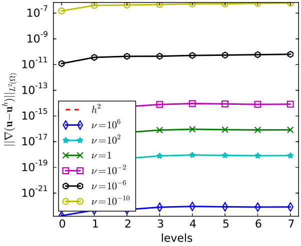
$$\boldsymbol{u} = \mathbf{0}, \quad p = 10\left((x-0.5)^3 y + (1-x)^2(y-0.5)^2 - \frac{1}{36}\right)$$

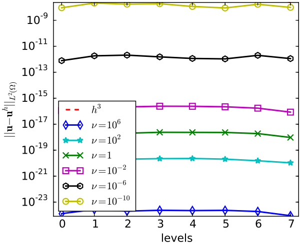
- Taylor–Hood P_2/P_1



  - velocity errors scale with inverse of viscosity

- Scott–Vogelius P_2/P_1^{disc}, barycentric-refined grid





  - round-off errors from high condition number of matrix of linear system of equations

- large velocity errors can occur in the standard situation $V_{\mathrm{div}}^h \not\subset V_{\mathrm{div}}$ in the presence of large pressure or small viscosity

- ways to implement finite elements
  - given triangulation of Ω with mesh cells $\{K\}$
  - unmapped finite elements
    - define the local finite element on the physical mesh cell K
  - mapped finite elements
    - define finite elements on a reference cell $\hat{K}$
    - define the finite element on K via the reference map from $\hat{K}$

- ways to implement finite elements
  - given triangulation of Ω with mesh cells $\{K\}$
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    - define the local finite element on the physical mesh cell K
  - mapped finite elements
    - define finite elements on a reference cell $\hat{K}$
    - define the finite element on K via the reference map from $\hat{K}$
- remarks
  - both ways gives the same results for affine reference maps (simplicial mesh cells, parallelepipeds)
  - mapped finite elements resemble a standard way for numerical analysis
  - mapped finite elements require the assembling of quadrature rules, degrees of freedom, nodal functionals only on reference cell

- our choice in the in-house code ParMooN [1,2]: mapped finite elements
  - reference maps are computed once and stored in a database
  - multi-linear reference maps possible
- many free libraries included
  - we found that PETSc is very helpful, since it includes itself many other libraries

[1] J., Matthies; Comput. Vis. Sci. 6, 163–169, 2004

[2] Wilbrandt, Bartsch, et al.; Comput. Math. Appl. 74, 74–88, 2017

- implementation
  - vector-valued velocity space

$$
\begin{aligned}
V^h &= \text{span}\{\boldsymbol{\phi}_i^h\}_{i=1}^{3N_v} \\
&= \text{span}\left\{\left\{\begin{pmatrix}\phi_i^h\\0\\0\end{pmatrix}\right\}_{i=1}^{N_v} \cup \left\{\begin{pmatrix}0\\\phi_i^h\\0\end{pmatrix}\right\}_{i=1}^{N_v} \cup \left\{\begin{pmatrix}0\\0\\\phi_i^h\end{pmatrix}\right\}_{i=1}^{N_v}\right\}
\end{aligned}
$$

  - pressure space

$$Q^h = \text{span}\{\psi_i^h\}_{i=1}^{N_p}$$

  - representation of unknown solution

$$\boldsymbol{u}^h = \sum_{j=1}^{3N_v} u_j^h \boldsymbol{\phi}_j^h, \quad p^h = \sum_{j=1}^{N_p} p_j^h \psi_j^h$$

- pressure finite element space
  - standard basis functions not in $L^2_0(\Omega)$
  - it can be shown under mild assumptions that standard basis functions can be used as ansatz and test functions
  - computed pressure with standard basis functions has to be projected into $L^2_0(\Omega)$ at the end

- linear saddle point problem

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

with

$$\begin{aligned} (A)_{ij} = a_{ij} &= \sum_{K \in \mathcal{T}^h} \left(\nabla \boldsymbol{\phi}_j^h, \nabla \boldsymbol{\phi}_i^h \right)_K, i,j = 1,\ldots,3N_v, \\ (B)_{ij} = b_{ij} &= -\sum_{K \in \mathcal{T}^h} \left(\nabla \cdot \boldsymbol{\phi}_j^h, \psi_i^h \right)_K, i = 1,\ldots,N_p, j = 1,\ldots,3N_v, \\ (\underline{f})_i = f_i &= \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{f}, \boldsymbol{\phi}_i^h \right)_K, i = 1,\ldots,3N_v \end{aligned}$$

- dimension (3d): $(3N_v + N_p) \times (3N_v + N_p)$

- matrix A
  - symmetric
  - positive definite
  - block-diagonal matrix

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

- matrix A
  - symmetric
  - positive definite
  - block-diagonal matrix

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

- $\left(\mathbb{D}\left(\boldsymbol{u}^h\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right)$ instead of $\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right)$
  - equivalent only if $\boldsymbol{u}^h$ weakly divergence-free
  - generally not given for finite element velocities
  - not longer block-diagonal matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix}$$

4. Stabilizing Non Inf-Sup Stable Finite Elements

- need implementation of different finite element spaces for velocity and pressure
- one obtains a linear algebraic saddle point problem

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

  - sparse direct solvers only efficient in 2d and for small and medium sized systems ($\lesssim 500\,000$ d.o.f.s)
  - construction of special preconditioners for iterative methods necessary, because of zero on main diagonal $\Longrightarrow$ Chapters 7 and 9

[1] J., Knobloch, Wilbrandt; book chapter in 'Fluids under Pressure', Springer, to appear 2019

- need implementation of different finite element spaces for velocity and pressure
- one obtains a linear algebraic saddle point problem

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

  - sparse direct solvers only efficient in 2d and for small and medium sized systems ($\lesssim 500\,000$ d.o.f.s)
  - construction of special preconditioners for iterative methods necessary, because of zero on main diagonal $\Longrightarrow$ Chapters 7 and 9
- goal: remove saddle point character by removing the zero block
  - introduce pressure-pressure coupling in mass balance
    - several proposals in literature, recent review in [1]
    - here: PSPG method
  - leads automatically to violation of mass conservation
  - enables use of same finite element spaces for velocity and pressure

[1] J., Knobloch, Wilbrandt; book chapter in 'Fluids under Pressure', Springer, to appear 2019

- Pressure Stabilization Petrov–Galerkin (PSPG) method [1]
- most popular method
- given $\boldsymbol{f} \in L^2(\Omega)$, find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$A_{\mathrm{pspg}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\mathrm{pspg}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \quad \forall \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$$

with

$$\begin{aligned} A_{\mathrm{pspg}}\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) &= \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) - \left(\nabla \cdot \boldsymbol{v}, p\right) + \left(\nabla \cdot \boldsymbol{u}, q\right) \\ &\quad + \sum_{E \in \mathcal{E}^h} \gamma_E \left([|p|]_E, [|q|]_E\right)_E + \sum_{K \in \mathcal{T}^h} \left(-\nu \Delta \boldsymbol{u} + \nabla p, \delta_K^p \nabla q\right)_K \end{aligned}$$

and

$$L_{\mathrm{pspg}}\left((\boldsymbol{v}, q)\right) = (\boldsymbol{f}, \boldsymbol{v}) + \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{f}, \delta_K^p \nabla q\right)_K$$

[1] Hughes, Franca, Balestra; Comput. Methods Appl. Mech. Engrg. 59, 85–99, 1986

- meaning of the terms

$$\sum_{E\in\mathcal{E}^h} \gamma_E \left([|p|]_E, [|q|]_E\right)_E + \sum_{K\in\mathcal{T}^h} \left(-\nu\Delta \boldsymbol{u} + \nabla p, \delta_K^p \nabla q\right)_K$$
$$\sum_{K\in\mathcal{T}^h} \left(\boldsymbol{f}, \delta_K^p \nabla q\right)_K$$

  - $\mathcal{E}^h$ – set of faces of mesh cells
  - E – face
  - $\mathcal{T}^h$ – triangulation, set of mesh cells
  - K – mesh cell
  - γ_E, δ_K^p – stabilization parameters, positive
  - $[|\cdot|]_E$ – jump across a face
- goal: appropriate choice of stabilization parameters by finite element error analysis

- finite element error analysis for stabilized methods is usually performed for norms that include stabilization
- norm for PSPG method

$$
\begin{aligned}
\left\| \left(\boldsymbol{v}^h, q^h \right) \right\|_{\mathrm{pspg}} &= \left(\nu \left\| \nabla \boldsymbol{v}^h \right\|_{L^2(\Omega)}^2 + \sum_{E \in \mathcal{E}^h} \gamma_E \left\| \left[\left| q^h \right|\right]_E \right\|_{L^2(E)}^2 \right. \\
&\qquad \left. + \sum_{K \in \mathcal{T}^h} \delta_K^p \left\| \nabla q^h \right\|_{L^2(K)}^2 \right)^{1/2}
\end{aligned}
$$

  - sum of seminorms
  - to check that from $\left\| \left(\boldsymbol{v}^h, q^h \right) \right\|_{\mathrm{pspg}} = 0$ it follows that $\boldsymbol{v}^h = \mathbf{0}$ and $q^h = 0$: direct calculation, board p. 202

- existence and uniqueness of a solution if

$$0 < \delta_K^p \le \frac{h_K^2}{\nu C_{\mathrm{inv}}^2} \quad \text{conditionally stable}$$

  - apply basic theorem of linear algebra (finite-dimensional problem):
    - bilinear coercive $\Longrightarrow$
    - corresponding matrix non-singular $\Longrightarrow$
    - unique solution for each right-hand side
    - show coercivity of bilinear form, board p. 202/203

$$A_{\mathrm{pspg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \ge \frac{1}{2}\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{pspg}}^2$$

- Definition: A stabilized discrete method is absolutely stable if it is stable for all $\delta > 0$, otherwise conditionally stable

- stability of solution

$$\left\|\left(\boldsymbol{u}^h, p^h\right)\right\|_{\mathrm{pspg}} \le 2\sqrt{2}\left(\frac{C}{\nu^{1/2}}\|\boldsymbol{f}\|_{L^2(\Omega)} + \left(\sum_{K\in\mathcal{T}^h}\delta_K^p\|\boldsymbol{f}\|_{L^2(K)}^2\right)^{1/2}\right)$$

  - use unique solution as test function

- stability of solution

$$\left\| \left(\boldsymbol{u}^h, p^h\right) \right\|_{\text{pspg}} \le 2\sqrt{2} \left(\frac{C}{\nu^{1/2}} \left\| \boldsymbol{f} \right\|_{L^2(\Omega)} + \left(\sum_{K \in \mathcal{T}^h} \delta_K^p \left\| \boldsymbol{f} \right\|_{L^2(K)}^2 \right)^{1/2} \right)$$

  - use unique solution as test function

- Galerkin orthogonality

$$A_{\text{pspg}} \left(\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h \right), \left(\boldsymbol{v}^h, q^h \right) \right) = 0 \quad \forall \ \left(\boldsymbol{v}^h, q^h \right) \in V^h \times Q^h$$

  - direct calculation using definition of PSPG method

- finite element error analysis: let

$$\delta_K^p = C_0 \frac{h_K^2}{\nu}, \quad \gamma_E = C_1 \frac{h_E}{\nu}$$

and

$$P_k \text{ or } Q_k \subseteq V^h \subset V, \quad k \geq 1, \qquad P_l \text{ or } Q_l \subseteq Q^h \subset Q, \quad l \geq 0,$$

then

$$\left\| \left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h \right) \right\|_{\text{pspg}} \leq C \left(\nu^{1/2} h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \frac{h^{l+1}}{\nu^{1/2}} \|p\|_{H^{l+1}(\Omega)} \right)$$

  - triangle inequality $\Longrightarrow$ interpolation error $+$ discrete term
  - coercivity for discrete term
  - estimate of individual terms on right-hand side by standard estimates (Cauchy–Schwarz, Young, interpolation error estimates) using bounds for stabilization parameters

- error estimate

$$\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|_{\text{pspg}} \le C\left(\nu^{1/2}h^{k}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + \frac{h^{l+1}}{\nu^{1/2}}\left\|p\right\|_{H^{l+1}(\Omega)}\right)$$

  - convergence in $\left\|\cdot\right\|_{\text{pspg}}$ is at least $\min\{k, l+1\}$
  - neglect all terms in $\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|_{\text{pspg}}$ but $\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}$ shows that ν^{-1} appears in front of the pressure term
  - for continuous pressure finite element spaces

$$\left\|\nabla\left(p-p^h\right)\right\|_{L^2(\Omega)} \le C\left(\nu h^{k-1}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + h^{l}\left\|p\right\|_{H^{l+1}(\Omega)}\right)$$

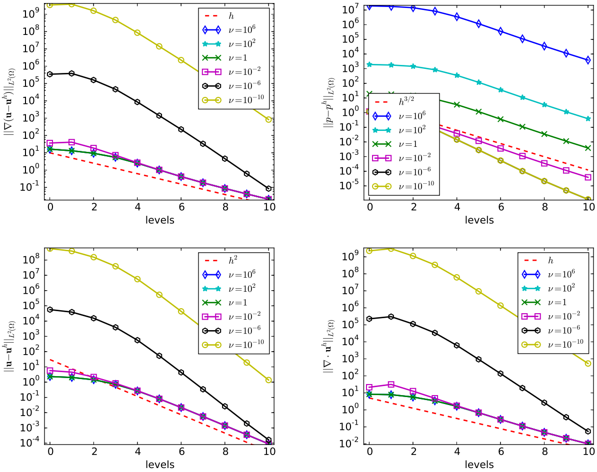
  - estimates for $\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)}$ and $\left\|p-p^h\right\|_{L^2(\Omega)}$ in [1]

[1] Brezzi, Douglas; Numer. Math. 53, 225–235, 1988

- analytical example which supports the error estimates
- prescribed solution

$$
\begin{aligned}
\boldsymbol{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix} = 200 \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix} \\
p &= 10 \left(\left(x - \frac{1}{2} \right)^3 y^2 + (1-x)^3 \left(y - \frac{1}{2} \right)^3 \right)
\end{aligned}
$$

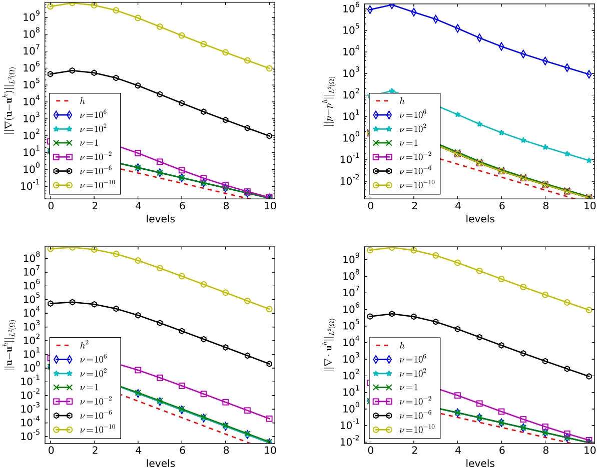
- P_1/P_1, $\delta_K^p = 0.5h_K^2/\nu$



  - larger velocity errors for small ν, but higher order of convergence

4 Numerical Example

- $P_1/P_0, \gamma_E = 0.5 h_E/\nu$



  - ○ larger velocity errors for small ν, but higher order of convergence

- construction of $\boldsymbol{u}^h_{\mathrm{RT}_0} \in \mathrm{RT}_0$ such that [1]

$$\left\|\nabla \cdot \left(\boldsymbol{u}^h + \boldsymbol{u}^h_{\mathrm{RT}_0}\right)\right\|_{L^2(\Omega)} = 0$$

weakly divergence-free

- correction

$$\boldsymbol{u}^h_{\mathrm{RT}_0} = \sum_{E\in\mathcal{E}^h} \frac{\gamma_E}{h_E} \left(\int_E [|p^h|]_E \; ds\right) \boldsymbol{\phi}_E$$

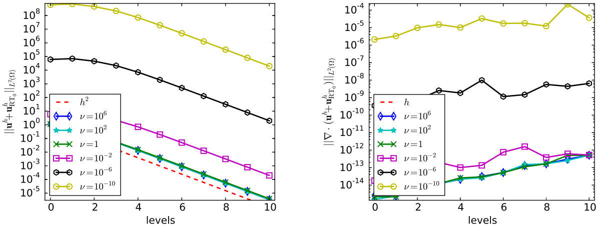
with RT_0 basis function

$$\boldsymbol{\phi}_E|_K = \pm\frac{h_E}{2\,|K|}\left(\boldsymbol{x} - \boldsymbol{x}_E\right) \in \mathrm{RT}_0(K)$$

can be computed locally

[1] Barrenechea, Valentin; Internat. J. Numer. Methods Engrg. 86, 801–815, 2011

- post-processed P_1/P_0



  - almost same velocity error in $L^2(\Omega)$
  - but solution is weakly divergence-free

- symmetric Galerkin Least Squares (GLS) method [1]
- find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$A_{\mathrm{gls}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\mathrm{gls}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \quad \forall \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h,$$

with

$$\begin{aligned}
&A_{\mathrm{gls}}\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) \\
&\quad = \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) - \left(\nabla \cdot \boldsymbol{v}, p\right) - \left(\nabla \cdot \boldsymbol{u}, q\right) \\
&\qquad - \sum_{E \in \mathcal{E}^h} \gamma_E \left(\left[|p|\right]_E, \left[|q|\right]_E\right)_E - \sum_{K \in \mathcal{T}^h} \left(-\nu \Delta \boldsymbol{u} + \nabla p, \delta_K^p \left(-\nu \Delta \boldsymbol{v} + \nabla q\right)\right)_K, \\
&L_{\mathrm{gls}}\left((\boldsymbol{v}, q)\right) = \left(\boldsymbol{f}, \boldsymbol{v}\right) - \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{f}, \delta_K^p \left(-\nu \Delta \boldsymbol{v} + \nabla q\right)\right)_K
\end{aligned}$$

- symmetric
- conditionally stable

[1] Hughes, Franca; Comput. Methods Appl. Mech. Engrg. 65, 85–96, 1987

- non-symmetric GLS method, absolutely stable method of Douglas and Wang [1]
- find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$A_{\mathrm{DW}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\mathrm{DW}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \quad \forall \ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h,$$

with

$$\begin{aligned} & A_{\mathrm{DW}}\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) \\ & \quad = \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) - \left(\nabla \cdot \boldsymbol{v}, p\right) + \left(\nabla \cdot \boldsymbol{u}, q\right) \\ & \qquad + \sum_{E \in \mathcal{E}^h} \gamma_E \left([|p|]_E, [|q|]_E\right)_E + \sum_{K \in \mathcal{T}^h} \left(-\nu\Delta\boldsymbol{u} + \nabla p, \delta_K^p\left(-\nu\Delta\boldsymbol{v} + \nabla q\right)\right)_K, \\ & L_{\mathrm{DW}}\left((\boldsymbol{v}, q)\right) = (\boldsymbol{f}, \boldsymbol{v}) + \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{f}, \delta_K^p\left(-\nu\Delta\boldsymbol{v} + \nabla q\right)\right)_K \end{aligned}$$

- non-symmetric
- replace q by $-q$: difference to symmetric GLS $+\nu\Delta\boldsymbol{v}$ instead of $-\nu\Delta\boldsymbol{v}$
- absolutely stable

[1] Douglas, Wang; Math. Comp. 52, 495–508, 1989

- framework for residual-based stabilizations in [1]
- absolutely stable modification of PSPG method proposed in [1]
  - using an alternative definition of the discrete Laplacian

$$\left(\Delta^h \boldsymbol{u}, \boldsymbol{v}^h\right) = -\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}^h\right) \quad \forall\, \boldsymbol{u} \in V,\ \boldsymbol{v}^h \in V^h$$

    requires solution of a system with mass matrix
- all methods identical if discrete Laplacian is not used, e.g., for P_1/P_1
- instability of symmetric GLS can be seen in numerical simulations, e.g., in [2]

[1] Bochev, Gunzburger; SIAM J. Numer. Anal. 42, 1189-1207, 2004

[2] J., Knobloch, Wilbrandt; book chapter in 'Fluids under Pressure', Springer, to appear 2019

- framework in [1]
- stabilization of Brezzi and Pitkäranta, oldest method [2]
- find $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h = P_1 \times P_1$ such that

$$
\begin{aligned}
\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) &= \left(\boldsymbol{f}, \boldsymbol{v}^h\right) && \forall\, \boldsymbol{v}^h \in V^h, \\
-\left(\nabla \cdot \boldsymbol{u}^h, q^h\right) - \sum_{K \in \mathcal{T}^h} \left(\nabla p^h, \delta_K^p \nabla q^h\right)_K &= 0 && \forall\, q^h \in Q^h
\end{aligned}
$$

- for P_1/P_1, same matrix is PSPG method

[1] Brezzi, Fortin; Numer. Math. 89, 457–491, 2001

[2] Brezzi, Pitkäranta; Notes Numer. Fluid Mech. 10, 11–19, 1984

- using fluctuations of the pressure [1]
- find $\left(\boldsymbol{u}^h, p^h, \overline{\nabla p^h}\right) \in V^h \times Q^h \times \overline{V^h}$ such that

$$
\begin{array}{rcll}
\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) & = & \left(\boldsymbol{f}, \boldsymbol{v}^h\right) & \forall\, \boldsymbol{v}^h \in V^h, \\
-\left(\nabla \cdot \boldsymbol{u}^h, q^h\right) - \sum\limits_{K \in \mathcal{T}^h} \left(\nabla p^h - \overline{\nabla p^h}, \delta_K^p \nabla q^h\right)_K & = & 0 & \forall\, q^h \in Q^h, \\
\left(\nabla p^h - \overline{\nabla p^h}, \overline{\boldsymbol{v}^h}\right) & = & 0 & \forall\, \boldsymbol{v}^h \in \overline{V^h}
\end{array}
$$

- other stabilizations with fluctuations of the pressure
  - local projection stabilization (LPS) [2], several variants meanwhile
  - method of Dohrmann and Bochev [3]
- stabilization with jumps across faces, continuous interior penalty (CIP) method [4]

[1] Codina, Blasco; Comput. Methods Appl. Mech. Engrg. 143, 373–391, 1997

[2] Becker, Braack; Calcolo 38, 173–199, 2001

[3] Dohrmann, Bochev; Internat. J. Numer. Methods Fluids 46, 183–201, 2004

[4] Burman, Hansbo; SIAM J. Numer. Anal. 44, 2393–2410, 2006

- usually extended matrix stencil of the pressure-pressure coupling matrix block C

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

- equal-order methods quite popular
  - often easy to implement
  - standard preconditioners in iterative solvers
- several proposals
  - not clear which is the best
  - numerical studies in [1]
    - PSPG, non-symmetric GLS, one variant of LPS behaved quite similarly
    - symmetric GLS shows instabilities if stabilization parameter is too large
- stabilization parameters
  - finite element error analysis gives asymptotic optimal choice
  - concrete choice depends on the user, optimal approach not known
- personal opinion: prefer inf-sup stable pairs of finite element spaces
  - benchmark problem for stationary Navier–Stokes equations [1]: Taylor–Hood more accurate than all stabilized methods

[1] J., Knobloch, Wilbrandt; book chapter in 'Fluids under Pressure', Springer, to appear 2019

5. On Mass Conservation and the Divergence Constraint

- continuous problem:

$$(\nabla \cdot \boldsymbol{u}, q) = 0 \quad \forall\, q \in Q$$

  - infinitely many conditions
  - inf-sup condition equivalent to $\nabla \cdot V = Q \Longrightarrow$ take $q = \nabla \cdot \boldsymbol{u}$

$$0 = \|\nabla \cdot \boldsymbol{u}\|_{L^2(\Omega)}$$ weakly divergence-free

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$$(\nabla \cdot \boldsymbol{u}, q) = 0 \quad \forall\, q \in Q$$

  - infinitely many conditions
  - inf-sup condition equivalent to $\nabla \cdot V = Q \Longrightarrow$ take $q = \nabla \cdot \boldsymbol{u}$

$$0 = \|\nabla \cdot \boldsymbol{u}\|_{L^2(\Omega)}$$ weakly divergence-free

- finite element problem

$$(\nabla \cdot \boldsymbol{u}^h, q^h) = 0 \quad \forall\, q^h \in Q^h$$ discretely divergence-free

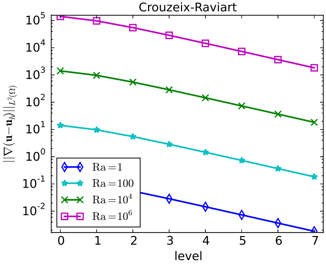
  - finite number of conditions
  - usually $\nabla \cdot V^h \not\subset Q^h$
  - $\Longrightarrow$ no mass conservation
    - not tolerable in certain applications

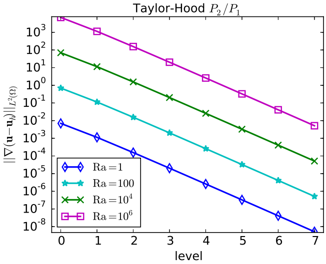
5 Numerical Example 1

- no-flow problem: prescribed solution for Stokes problem with $\nu = 1$

$$\boldsymbol{u} = \mathbf{0}, \quad p = \mathrm{Ra}\left(y^3 - \frac{y^2}{2} + y - \frac{7}{12}\right)$$

- finite elements
  - P_2/P_1 – Taylor–Hood
  - P_1^{nc}/P_0 – Crouzeix–Raviart



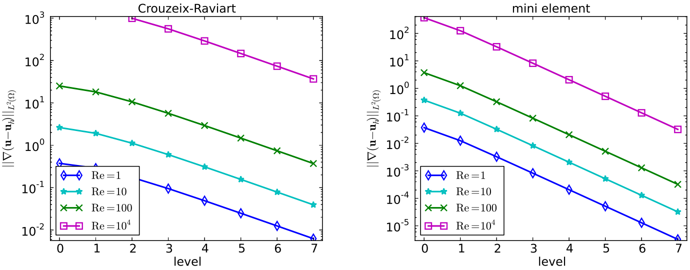


  - velocity error scales with the pressure

- stationary vortex: prescribed solution for Navier–Stokes problem with $\nu = 1$

$$\boldsymbol{u} = \begin{pmatrix} -y \\ x \end{pmatrix}, \quad p = \mathrm{Re}\left(\frac{x^2+y^2}{2} - \frac{1}{3}\right), \quad \mathrm{Re} > 0$$

- balance of nonlinear term of the Navier–Stokes equations and pressure term
  - P_1^{bubble}/P_1 – MINI element



  - velocity error scales with the pressure

5 Conclusions

- error scales also with Coriolis force if this term is present
  - important in meteorology

- error scales also with Coriolis force if this term is present
  - important in meteorology
- fundamental invariance property for continuous equations (with Dirichlet boundary conditions)

$$\boldsymbol{f} \to \boldsymbol{f} + \nabla\psi \quad \Longrightarrow \quad (\boldsymbol{u}, p) \to (\boldsymbol{u}, p + \psi)$$

  physically correct behavior
  - obviously not satisfied for the considered discretizations
    - no-flow problem: $\boldsymbol{f} = \nabla p$
    - but change of $\boldsymbol{f}$ changed $\boldsymbol{u}^h$

    unphysical behavior
  - connected with Helmholtz decomposition of vector fields in $L^2(\Omega)$: every vector field in $L^2(\Omega)$ can be decomposed into a gradient field and a weakly divergence-free field

- very popular technique
- add in continuous problem

$$\mathbf{0} = -\mu\nabla(\nabla\cdot\boldsymbol{u})$$

- gives in finite element problem

$$\mu(\nabla\cdot\boldsymbol{u}^h, \nabla\cdot\boldsymbol{v}^h)$$

  does not vanish if $\nabla\cdot\boldsymbol{u}^h \neq 0$
- stabilized finite element Stokes problem: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$\begin{array}{rcll}\nu\left(\nabla\boldsymbol{u}^h, \nabla\boldsymbol{v}^h\right) - \left(\nabla\cdot\boldsymbol{v}^h, p^h\right) & & & \\ + \displaystyle\sum_{K\in\mathcal{T}^h}\mu_K\left(\nabla\cdot\boldsymbol{u}^h, \nabla\cdot\boldsymbol{v}^h\right)_K & = & \langle\boldsymbol{f}, \boldsymbol{v}^h\rangle_{V',V} & \forall\, \boldsymbol{v}^h \in V^h, \\ -\left(\nabla\cdot\boldsymbol{u}^h, q^h\right) & = & 0 & \forall\, q^h \in Q^h\end{array}$$

$\{\mu_K\}$ with $\mu_K \geq 0$ – stabilization parameters

- fixed triangulation:

$$\lim_{\mu_{\min}\to\infty} \left\|\nabla\cdot\boldsymbol{u}^h\right\|_{L^2(\Omega)} = 0$$

  - Proof: consider $\mu_{\min} = \mu = \mu_K$ for all $K \in \mathcal{T}^h$
    - use $\boldsymbol{u}^h$ as test function, dual pairing, Young's inequality

$$\begin{aligned} &\nu\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)}^2 + \mu\left\|\nabla\cdot\boldsymbol{u}^h\right\|_{L^2(\Omega)}^2 \\ &= \quad \langle\boldsymbol{f},\boldsymbol{u}^h\rangle_{V',V} \le \|\boldsymbol{f}\|_{H^{-1}(\Omega)}\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)} \\ &\le \quad \frac{1}{4\nu}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}^2 + \nu\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)}^2 \end{aligned}$$

    - consequently

$$\left\|\nabla\cdot\boldsymbol{u}^h\right\|_{L^2(\Omega)} \le \frac{1}{2\nu^{1/2}\mu^{1/2}}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}$$

    - statement follows

- Question: Large stabilization parameters good for other errors ?

- answer by finite element error analysis
- different situations can be distinguished, see [1]
- standard situation leads to estimate

$$\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}^2 \leq \inf_{\boldsymbol{v}^h \in V_{\mathrm{div}}^h}\left(4\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(\Omega)}^2+2\frac{\mu}{\nu}\left\|\nabla \cdot \boldsymbol{v}^h\right\|_{L^2(\Omega)}^2\right) + \frac{2}{\mu\nu} \inf_{q^h \in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}^2$$

  - present at board, p. 223
  - velocity error (bound) depends still on pressure
  - dependency on viscosity is $\nu^{-1/2}$ instead of ν^{-1} without grad-div stabilization

[1] Jenkins, J., Linke, Rebholz; Adv. Comput. Math. 40, 491–516, 2014

- concrete estimate for Taylor–Hood pair of finite element spaces $V^h/Q^h = P_k/P_{k-1}, k \geq 2$

$$
\begin{aligned}
\left\| \nabla \left(\boldsymbol{u} - \boldsymbol{u}^h \right) \right\|_{L^2(\Omega)}^2 \quad &\leq \quad \left(4 + \frac{2\mu}{\nu} \right) C_{V_{\mathrm{div}}^h}^2 h^{2k} \left\| \boldsymbol{u} \right\|_{H^{k+1}(\Omega)}^2 \\
&\quad + \frac{2C_{Q^h}^2}{\mu\nu} h^{2k} \left\| p \right\|_{H^k(\Omega)}^2
\end{aligned}
$$

  - optimal choice of stabilization parameter by minimizing error bound
  - depends on unknown norms of the solution
  - depends on unknown constants for the best approximation error estimates
  - if one assumes that all unknown quantities are $\sim 1 \Longrightarrow \mu \sim 1$
- similar considerations for MINI element: $\mu \sim h$
- optimal stabilization parameter for velocity error is not large

- summary
  - lot of experience in literature with grad-div stabilization and parameters that are appropriate for good error bounds
    - improves mass conservation somewhat, but usually not essential
    - velocity error still depends on pressure
  - grad-div term leads to matrix block

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix} \quad \text{instead of} \quad \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

  - grad-div stabilization is not the solution of the problem

- stability and mass conservation in finite element methods are conflicting requirements
  - let V^h be fixed
  - stability: discrete inf-sup condition

$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{(\nabla \cdot \boldsymbol{v}^h, p^h)}{\|\nabla \boldsymbol{v}^h\|_{L^2(\Omega)} \|q^h\|_{L^2(\Omega)}} \geq \beta_{\text{is}}^h > 0$$

given if Q^h is sufficiently small

- stability and mass conservation in finite element methods are conflicting requirements
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$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\mathbf{0}\}} \frac{(\nabla \cdot \boldsymbol{v}^h, p^h)}{\|\nabla \boldsymbol{v}^h\|_{L^2(\Omega)} \|q^h\|_{L^2(\Omega)}} \geq \beta_{\text{is}}^h > 0$$

    given if Q^h is sufficiently small

  - mass conservation: $\nabla \cdot V^h \subseteq Q^h$: take $q^h = \nabla \cdot \boldsymbol{u}^h$

$$0 = -\left(\nabla \cdot \boldsymbol{u}^h, q^h\right) = -\left\|\nabla \cdot \boldsymbol{u}^h\right\|_{L^2(\Omega)}^2$$

    given if Q^h is sufficiently large

- stability and mass conservation in finite element methods are conflicting requirements
  - let V^h be fixed
  - stability: discrete inf-sup condition

$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{(\nabla \cdot \boldsymbol{v}^h, p^h)}{\|\nabla \boldsymbol{v}^h\|_{L^2(\Omega)} \|q^h\|_{L^2(\Omega)}} \geq \beta_{\text{is}}^h > 0$$

    given if Q^h is sufficiently small
  - mass conservation: $\nabla \cdot V^h \subseteq Q^h$: take $q^h = \nabla \cdot \boldsymbol{u}^h$

$$0 = -\left(\nabla \cdot \boldsymbol{u}^h, q^h\right) = -\left\|\nabla \cdot \boldsymbol{u}^h\right\|_{L^2(\Omega)}^2$$

    given if Q^h is sufficiently large
- $\nabla \cdot V^h \subset L^2(\Omega) \Longleftrightarrow$ normal components of finite element functions are continuous
  - note: not satisfied for Crouzeix–Raviart finite element

- analytical tool: smooth de Rham complex or Stokes complex in two dimensions

$$\mathbb{R} \quad \to \quad H^2(\Omega) \quad \overset{\mathbf{curl}}{\to} \quad H^1(\Omega) \quad \overset{\text{div}}{\to} \quad L^2(\Omega) \quad \to \quad 0$$

  with

$$\mathbf{curl}\ v(\boldsymbol{x}) = \begin{pmatrix} -\partial_y v \\ \partial_x v \end{pmatrix} (\boldsymbol{x})$$

  - sequence of spaces and maps
  - de Rham complex is called exact if the range of each operator is the kernel of the succeeding operator
    - if $w \in H^2(\Omega)$ is curl-free, then w is constant function
    - if $\boldsymbol{v} \in H^1(\Omega)$ is divergence-free, then $\boldsymbol{v} = \mathbf{curl}\ w$ for some $w \in H^2(\Omega)$
    - the map div $: H^1(\Omega) \to L^2(\Omega)$ is surjective, since the kernel of the last operator is $L^2(\Omega)$

- finite element sub-complex

$$\mathbb{R} \quad \to \quad W^h \quad \overset{\mathbf{curl}}{\to} \quad V^h \quad \overset{\text{div}}{\to} \quad Q^h \quad \to \quad 0$$

  - if finite element sub-complex is exact
    - V^h/Q^h satisfies the discrete inf-sup condition
    - weakly divergence-free velocity fields are computed, since div $V^h = Q^h$

    goal: construction of exact finite element sub-complex

[1] Ciarlet; The finite element method for elliptic problems, 1978, Chapter 6.1

- finite element sub-complex

$$\mathbb{R} \quad \to \quad W^h \quad \stackrel{\mathbf{curl}}{\to} \quad V^h \quad \stackrel{\text{div}}{\to} \quad Q^h \quad \to \quad 0$$

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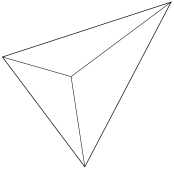
    - V^h/Q^h satisfies the discrete inf-sup condition
    - weakly divergence-free velocity fields are computed, since div $V^h = Q^h$

  goal: construction of exact finite element sub-complex

- example
  - consider barycentric refinement of triangles
  - Hsieh–Clough–Tocher finite element [1]
  - composite element of third order polynomials on each fine mesh cell
  - requirement: continuously differentiable $\Longrightarrow$ finite element space W^h belongs to $H^2(\Omega)$



[1] Ciarlet; The finite element method for elliptic problems, 1978, Chapter 6.1

- example (cont.)
  - differentiation reduces the polynomial degree by one and also the regularity by one
  - $\Longrightarrow V^h = \mathbf{curl}\, W^h \subset H^1(\Omega)$ and $V^h = P_2$ (on barycentric-refined mesh)
  - $\Longrightarrow Q^h =$ div $V^h \subset L^2(\Omega)$ and $Q^h = P_1^{\mathrm{disc}}$ (on barycentric-refined mesh)

[1] Scott, Vogelius; in Large-scale computations in fluid mechanics, Part 2, 221–244, 1985

- example (cont.)
  - differentiation reduces the polynomial degree by one and also the regularity by one
  - $\Longrightarrow V^h = \mathbf{curl}\, W^h \subset H^1(\Omega)$ and $V^h = P_2$ (on barycentric-refined mesh)
  - $\Longrightarrow Q^h =$ div $V^h \subset L^2(\Omega)$ and $Q^h = P_1^{\mathrm{disc}}$ (on barycentric-refined mesh)
  - to show exactness: div $: V^h \to Q^h$ is a surjection
    - finite-dimensional spaces: by counting the number of degrees of freedom (somewhat longer)
  - P_2/P_1^{disc} – Scott–Vogelius pair of spaces [1] on barycentric refined grids is stable and computes weakly divergence-free solutions

[1] Scott, Vogelius; in Large-scale computations in fluid mechanics, Part 2, 221–244, 1985

- similar constructions can be started with other $H^2(\Omega)$ conforming finite element spaces
  - lead to high polynomial spaces, of little importance in practice
- situation in two dimensions more or less clear

[1] Zhang; Math. Comp. 74, 543–554, 2005

- similar constructions can be started with other $H^2(\Omega)$ conforming finite element spaces
  - lead to high polynomial spaces, of little importance in practice
- situation in two dimensions more or less clear
- 3d case much more challenging
  - possible de Rham complex

$$\mathbb{R} \to H^2(\Omega) \stackrel{\mathbf{grad}}{\to} H^1(\mathbf{curl};\Omega) \stackrel{\mathbf{curl}}{\to} H^1(\Omega) \stackrel{\mathrm{div}}{\to} L^2(\Omega) \to 0$$

    leads to velocity space with polynomials of degree 6
  - Scott–Vogelius pair of spaces $P_k/P_{k-1}^{\mathrm{disc}}$ is stable on barycentric refined meshes for $k \geq 3$, [1]

[1] Zhang; Math. Comp. 74, 543–554, 2005

- summary
  - Scott–Vogelius pair of spaces $P_k/P_{k-1}^{\mathrm{disc}}$, $k \ge d$, so far only pair that is used sometimes
  - little hope to construct any other lower order pair

- abandon conformity of finite element velocity space: $V^h \not\subset V$
- require $\nabla \cdot V^h \subset L^2(\Omega) \Longrightarrow$ study $H\,(\mathrm{div},\Omega)$ conforming finite elements
  - Raviart–Thomas elements, BDM elements
  - normal component of functions is continuous across faces
- difficulty: consistency error in discretizing the viscous term $(\nu\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$
  - no convergence for using just

$$\sum_{K\in\mathcal{T}^h} \int_K \nu\nabla \boldsymbol{u}^h : \nabla \boldsymbol{v}^h \, d\boldsymbol{x}$$

- proposals
  - modify bilinear form
  - modify $H\,(\mathrm{div},\Omega)$ to impose tangential continuity in a weak sense

- modify bilinear form: possible proposal

$$\sum_{K\in\mathcal{T}^h} \int_K \nabla \boldsymbol{u}^h : \nabla \boldsymbol{v}^h \, d\boldsymbol{x} - \sum_{E\in\mathcal{E}^h} \left(\int_E \{\{\varepsilon(\boldsymbol{u}^h)\}\}_E \, [|\boldsymbol{v}^h|]_E \, d\boldsymbol{s} \right.$$
$$\left. + \int_E \{\{\boldsymbol{v}^h\}\}_E \, [|\boldsymbol{u}^h|]_E \, d\boldsymbol{s} - \frac{\sigma}{h_E} \int_E [|\boldsymbol{u}^h|]_E \, [|\boldsymbol{v}^h|]_E \, d\boldsymbol{s} \right)$$

  - $[|\cdot|]_E$ – jump across face
  - $\{\{\cdot\}\}_E$ – average on face
  - $\{\{\varepsilon\cdot\}\}_E$ – average of tangential component on face
  - σ – parameter
  - reminds on DG

- modify $H(\mathrm{div}, \Omega)$ to impose tangential continuity in a weak sense
  - local space (2d)

$$\hat{V}^h(K) = V^h(K) + \mathbf{curl}\,(b_K S(K))$$

    - b_K – bubble function
    - $S(K)$ – auxiliary space
  - example: $V^h(K) = \mathrm{RT}_0(K)$, $S(K) = P_1(K)$
    - global space possesses correct order of consistency error
    - optimal error estimates can be proved

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    - global space possesses correct order of consistency error
    - optimal error estimates can be proved
- summary
  - use of $H\,(\mathrm{div}, \Omega)$-conforming methods interesting option
  - so far no own experience

- idea: satisfy discrete version of fundamental invariance principle by using appropriate reconstructions of the test functions [1]
- finite element Stokes problem with reconstruction: Given $\boldsymbol{f} \in L^2(\Omega)$, find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$
\begin{array}{rcll}
\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) & = & \left(\boldsymbol{f}, \Pi^h \boldsymbol{v}^h\right) & \forall\, \boldsymbol{v}^h \in V^h, \\
-\left(\nabla \cdot \boldsymbol{u}^h, q^h\right) & = & 0 & \forall\, q^h \in Q^h
\end{array}
$$

with

$$
\Pi^h \,:\, V^h \to R^h \quad (R^h - H\left(\mathrm{div}, \Omega\right)\text{-conforming fe space})
$$

[1] Linke; Comput. Methods Appl. Mech. Engrg. 268, 782–800, 2014

- consider $V^h \times Q^h = P_2^{\mathrm{bubble}} \times P_1^{\mathrm{disc}}$
  - discontinuous pressure of importance for easy construction
  - $R^h = \mathrm{RT}_1$
  - general requirements on Π^h: projection and interpolation properties

$$\begin{aligned}
\int_K (\boldsymbol{v} - \Pi^h \boldsymbol{v})\, d\boldsymbol{x} &= 0, \quad \forall\, \boldsymbol{v} \in V, \forall\, K \in \mathcal{T}^h \\
\int_E (\boldsymbol{v} - \Pi^h \boldsymbol{v}) \cdot \boldsymbol{n}_E q^h \, d\boldsymbol{s} &= 0, \quad \forall\, \boldsymbol{v} \in V, \forall\, q^h \in P_1(E) \\
\left\| \Pi^h \boldsymbol{v} - \boldsymbol{v} \right\|_{L^2(K)} &\le C h_K^m \left| \boldsymbol{v} \right|_{H^m(K)}, \quad m = 0, 1, 2
\end{aligned}$$

- cell-wise computation of $\Pi^h \boldsymbol{v}^h$ possible

- $V^h \times Q^h = P_2^{\text{bubble}} \times P_1^{\text{disc}}$ (cont.)
  - it holds for all $K \in \mathcal{T}^h$, $\boldsymbol{v} \in V$, and $q^h \in Q^h$

$$\begin{aligned}
\int_K \nabla \cdot \boldsymbol{v} q^h \, d\boldsymbol{x} &\overset{\text{prod. rule}}{=} \int_K \nabla \cdot (\boldsymbol{v} q^h) \, d\boldsymbol{x} - \int_K \nabla q^h \cdot \boldsymbol{v} \, d\boldsymbol{x} \\
&\overset{\text{int. by parts}}{=} \int_{\partial K} q^h \boldsymbol{v} \cdot \boldsymbol{n}_T \, d\boldsymbol{s} - \int_K \nabla q^h \cdot \boldsymbol{v} \, d\boldsymbol{x} \\
&\overset{\text{prop. 1 \& 2}}{=} \int_{\partial K} q^h (\Pi^h \boldsymbol{v}) \cdot \boldsymbol{n}_T \, d\boldsymbol{s} - \int_K \nabla q^h \cdot (\Pi^h \boldsymbol{v}) \, d\boldsymbol{x} \\
&\overset{\text{int. by parts}}{=} \int_K \nabla \cdot (\Pi^h \boldsymbol{v}) q^h \, d\boldsymbol{x}
\end{aligned}$$

$\Longrightarrow$ Q^h projection of the divergence $=$ Q^h projection of the divergence of the Π^h projection

- $V^h \times Q^h = P_2^{\text{bubble}} \times P_1^{\text{disc}}$ (cont.)
  - fundamental invariance principle for modified discretization:
    for all $\boldsymbol{v}^h \in V_{\text{div}}^h$ ($\Longrightarrow$ consider only velocity)

$$
\begin{aligned}
\left(\boldsymbol{f} + \nabla\left(P_{Q^h}\psi\right), \Pi^h \boldsymbol{v}^h\right) &\overset{\text{int. by parts}}{=} \left(\boldsymbol{f}, \Pi^h \boldsymbol{v}^h\right) - \left(\nabla \cdot \left(\Pi^h \boldsymbol{v}^h\right), P_{Q^h}\psi\right) \\
&\overset{\text{div. prop}}{=} \left(\boldsymbol{f}, \Pi^h \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, P_{Q^h}\psi\right) \\
&\overset{\text{disc. div-free}}{=} \left(\boldsymbol{f}, \Pi^h \boldsymbol{v}^h\right)
\end{aligned}
$$

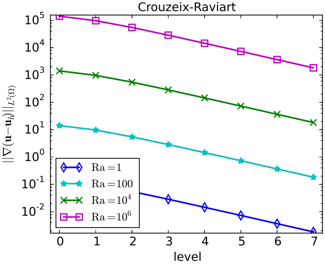
discrete counterpart of fundamental invariance principle satisfied: $\nabla\left(P_{Q^h}\psi\right)$ possesses no impact on finite element velocity

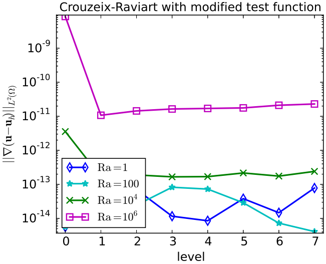
- finite element error analysis
  - consistency error is introduced, is of optimal order
  - estimate for velocity error

$$\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)} \leq 2(1+C_{\mathrm{PF}}) \inf_{\boldsymbol{v}^h \in V^h} \left\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\right\|_{L^2(\Omega)} + Ch^2 \left|\boldsymbol{u}\right|_{H^3(\Omega)}$$

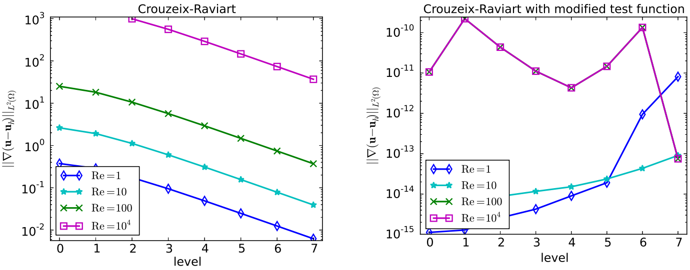
does not depend on pressure

- Example 1
  - Crouzeix–Raviart finite element with reconstruction in RT_0





- Example 2
  - Crouzeix–Raviart finite element with reconstruction in RT_0



- summary
  - extensions to almost arbitrary finite element pairs with discontinuous pressure in [1] (simplicial and brick-shaped meshes)
  - extension to finite element pairs with continuous pressure [2], becomes more complicated
  - extension to Navier–Stokes equations possible (and partly research in progress)
    - modifications of nonlinear convective term and of term with temporal derivative necessary
    - leads to modifications of the system matrix
  - divergence-free finite element velocity in R^h by applying reconstruction operator to $\boldsymbol{u}^h$

[1] Linke, Matthies, Tobiska; ESAIM: M2AN 50, 289–309, 2016

[2] Lederer, Linke, Merdon Schöberl; SIAM J. Numer. Anal. 55, 1291–1314, 2017

5 Summary of this Part

- most standard pairs of finite element spaces do not preserve mass
- grad-div stabilization does not solve the problem
- only conforming divergence-free stable pair that is of some importance is Scott–Vogelius pair
- $H(\mathrm{div}, \Omega)$-conforming finite element methods very interesting option, but no own experience so far
- alternative approach: pressure robust methods with appropriate test functions by reconstructions in $H(\mathrm{div}, \Omega)$-conforming finite element spaces
  - several new developments in this direction, check the publications of A. Linke (WIAS)

- survey of Part 5 in [1,2]

[1] J., Linke, Merdon, Neilan, Rebholz; SIAM Review 59, 492–544, 2017

[2] J.: *Finite Element Methods for Incompressible Flow Problems* 2016

6. Stabilizing Dominant Convection for Oseen Problems

- continuous equation

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + c\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

for simplicity: homogeneous Dirichlet boundary conditions

- difficulties:
  - coupling of velocity and pressure
  - dominating convection
- properties
  - linear

Carl Wilhelm Oseen (1879 – 1944)

- coefficients
  - $\nu > 0$
  - $\boldsymbol{b} \in W^{1,\infty}(\Omega)$, $\nabla \cdot \boldsymbol{b} = 0$
  - $c \in L^{\infty}(\Omega)$, $c(\boldsymbol{x}) \geq c_0 \geq 0$

- coefficients
  - $\nu > 0$
  - $\boldsymbol{b} \in W^{1,\infty}(\Omega)$, $\nabla \cdot \boldsymbol{b} = 0$
  - $c \in L^{\infty}(\Omega)$, $c(\boldsymbol{x}) \geq c_0 \geq 0$
- scaling of momentum equation:
  - $\|\boldsymbol{b}\|_{L^{\infty}(\Omega)} \sim 1$ if $\nu \leq \|\boldsymbol{b}\|_{L^{\infty}(\Omega)}$

- coefficients
  - $\nu > 0$
  - $\boldsymbol{b} \in W^{1,\infty}(\Omega)$, $\nabla \cdot \boldsymbol{b} = 0$
  - $c \in L^{\infty}(\Omega)$, $c(\boldsymbol{x}) \geq c_0 \geq 0$
- scaling of momentum equation:
  - $\|\boldsymbol{b}\|_{L^{\infty}(\Omega)} \sim 1$ if $\nu \leq \|\boldsymbol{b}\|_{L^{\infty}(\Omega)}$
- interesting cases
  - ν of moderate size, $c = 0$

    in numerical solution of steady-state Navier–Stokes equations
  - ν of arbitrary size, $c \sim (\Delta t)^{-1}$

    in numerical solution of time-dependent Navier–Stokes equations

- weak form

$$\begin{array}{rcll} \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + c\boldsymbol{u}, \boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, p) & = & \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V',V} & \forall\ \boldsymbol{v}\in V, \\ -(\nabla\cdot\boldsymbol{u}, q) & = & 0 & \forall\ q\in Q \end{array}$$

- bilinear forms

$$\begin{array}{llcl} a\ :\ V\times V\to\mathbb{R}, & a(\boldsymbol{u},\boldsymbol{v}) & = & \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + c\boldsymbol{u}, \boldsymbol{v}), \\ b\ :\ V\times Q\to\mathbb{R}, & b(\boldsymbol{v},q) & = & -(\nabla\cdot\boldsymbol{v}, q) \end{array}$$

- weak form

$$
\begin{aligned}
\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + c\boldsymbol{u}, \boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, p) &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{V',V} \quad \forall\, \boldsymbol{v}\in V,\\
-(\nabla\cdot\boldsymbol{u}, q) &= 0 \qquad\qquad\quad \forall\, q \in Q
\end{aligned}
$$

- bilinear forms

$$
\begin{aligned}
a\ :\ V\times V \to \mathbb{R}, \qquad a(\boldsymbol{u},\boldsymbol{v}) &= \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + c\boldsymbol{u}, \boldsymbol{v}),\\
b\ :\ V\times Q \to \mathbb{R}, \qquad b(\boldsymbol{v},q) &= -(\nabla\cdot\boldsymbol{v}, q)
\end{aligned}
$$

- existence and uniqueness of solution
  - proof: board, p. 246
  - essential condition

$$
((\boldsymbol{b}\cdot\nabla)\boldsymbol{v}, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v}\in V
$$

can be proved if $\boldsymbol{b}$ is divergence-free, board p. 245

- stability of solution
  - dependency of bounds on coefficients is important
  - depending on regularity of data, different estimates possible
    - most general

$$\frac{\nu}{2}\left\|\nabla \boldsymbol{u}\right\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}^2 \le \frac{1}{2\nu}\left\|\boldsymbol{f}\right\|_{H^{-1}(\Omega)}^2$$

    - $\boldsymbol{f} \in L^2(\Omega)$ and $c_0 > 0$

$$\nu\left\|\nabla \boldsymbol{u}\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}^2 \le \frac{1}{2c_0}\left\|\boldsymbol{f}\right\|_{L_2(\Omega)}^2$$

  - proof: board, p. 247

- stability of solution
  - dependency of bounds on coefficients is important
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$$\frac{\nu}{2}\left\|\nabla \boldsymbol{u}\right\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}^2 \le \frac{1}{2\nu}\left\|\boldsymbol{f}\right\|_{H^{-1}(\Omega)}^2$$

    - $\boldsymbol{f} \in L^2(\Omega)$ and $c_0 > 0$

$$\nu\left\|\nabla \boldsymbol{u}\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}^2 \le \frac{1}{2c_0}\left\|\boldsymbol{f}\right\|_{L_2(\Omega)}^2$$

  - proof: board, p. 247
  - estimates for pressure with inf-sup condition

- Galerkin finite element method

$$\begin{array}{rcll} a\left(\boldsymbol{u}^h, \boldsymbol{v}^h\right) + b\left(\boldsymbol{v}^h, p^h\right) & = & \left(\boldsymbol{f}, \boldsymbol{v}^h\right) & \forall\, \boldsymbol{v}^h \in V^h, \\ b\left(\boldsymbol{u}^h, q^h\right) & = & 0 & \forall\, q^h \in Q^h \end{array}$$

  - homogeneous Dirichlet boundary conditions
  - conforming, inf-sup stable finite element spaces
- existence, uniqueness, stability like for continuous problem

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$, bounded, polyhedral, Lipschitz-continuous boundary
  - regularity of coefficients like stated above

$$
\begin{aligned}
&\nu^{1/2} \left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} + \left\|c^{1/2}\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} \\
&\quad \le C\left[\left(1+\frac{1}{\beta_{\mathrm{is}}^h}\right) C_{\mathrm{os}} \inf_{\boldsymbol{v}^h\in V^h} \left\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\right\|_{L^2(\Omega)} + \frac{1}{\nu^{1/2}} \inf_{q^h\in Q^h} \left\|p-q^h\right\|_{L^2(\Omega)}\right]
\end{aligned}
$$

where

$$
C_{\mathrm{os}} = \nu^{1/2} + \|c\|_{L^\infty(\Omega)}^{1/2} + \|\boldsymbol{b}\|_{L^\infty(\Omega)} \min\left\{\frac{1}{\nu^{1/2}}, \frac{1}{c_0^{1/2}}\right\}
$$

  - C does not depend on coefficients and triangulation, but on Ω (Poincaré–Friedrichs inequality)

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity (cont.)
  - proof: principally same as for Stokes equations
  - estimates for convective term

$$
\begin{aligned}
\left|\left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\eta},\boldsymbol{\phi}^h\right)\right| &= \left|-\left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\phi}^h,\boldsymbol{\eta}\right)\right| \le \|\boldsymbol{b}\|_{L^\infty(\Omega)}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}\|\boldsymbol{\eta}\|_{L^2(\Omega)} \\
&\le \frac{2}{\nu}\|\boldsymbol{b}\|_{L^\infty(\Omega)}^2\|\boldsymbol{\eta}\|_{L^2(\Omega)}^2+\frac{\nu}{8}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2
\end{aligned}
$$

or if $c_0 > 0$

$$
\begin{aligned}
\left|\left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\eta},\boldsymbol{\phi}^h\right)\right| &\le \left\|c^{-1/2}\boldsymbol{b}\right\|_{L^\infty(\Omega)}\|\nabla\boldsymbol{\eta}\|_{L^2(\Omega)}\left\|c^{1/2}\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\
&\le \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}^2\|\nabla\boldsymbol{\eta}\|_{L^2(\Omega)}^2}{c_0}+\frac{\left\|c^{1/2}\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2}{4}
\end{aligned}
$$

- finite element error estimate for the $L^2(\Omega)$ norm of the pressure
  - same assumptions as for previous estimate

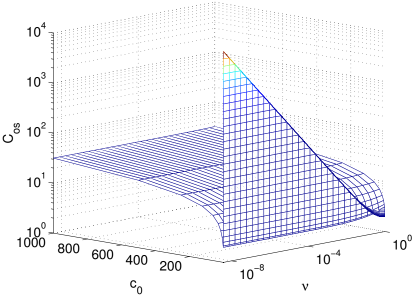
$$
\begin{aligned}
\left\|p - p^h\right\|_{L^2(\Omega)} \quad \le \quad & C\left[\frac{1}{\beta_{\mathrm{is}}^h}\left(1+\frac{1}{\beta_{\mathrm{is}}^h}\right) C_{\mathrm{os}}^2 \inf_{\boldsymbol{v}^h\in V^h}\left\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\right\|_{L^2(\Omega)}\right. \\
& \left. +\left(1+\frac{1}{\beta_{\mathrm{is}}^h}+\frac{1}{\beta_{\mathrm{is}}^h}\frac{C_{\mathrm{os}}}{\nu^{1/2}}\right)\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\right]
\end{aligned}
$$

- proof: as for Stokes equations, with discrete inf-sup condition

- finite element error estimates for conforming pairs of finite element spaces
  - same assumptions on domain as for previous estimates
  - solution sufficiently regular
  - h – mesh width of triangulation
  - spaces
    - $P_k^{\mathrm{bubble}}/P_k$, $k = 1$ (MINI element),
    - P_k/P_{k-1}, Q_k/Q_{k-1}, $k \geq 2$ (Taylor–Hood element),
    - $P_k^{\mathrm{bubble}}/P_{k-1}^{\mathrm{disc}}$, $Q_k/P_{k-1}^{\mathrm{disc}}$, $k \geq 2$

$$
\begin{aligned}
\left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)} &\leq \frac{C}{\nu^{1/2}} h^k \left(C_{\mathrm{os}} \left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + \frac{1}{\nu^{1/2}} \left\|p\right\|_{H^k(\Omega)} \right), \\
\left\|p - p^h\right\|_{L^2(\Omega)} &\leq C h^k \left(C_{\mathrm{os}}^2 \left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + \left(1 + \frac{C_{\mathrm{os}}}{\nu^{1/2}} \right) \left\|p\right\|_{H^k(\Omega)} \right)
\end{aligned}
$$

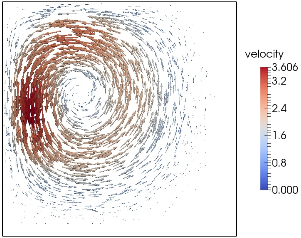
- $C_{\rm os}$ for $\|\boldsymbol{b}\|_{L^\infty(\Omega)} = 1$

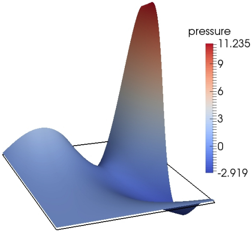


- error bounds not uniform for small ν or small time steps

- analytical example which supports the error estimates
- prescribed solution

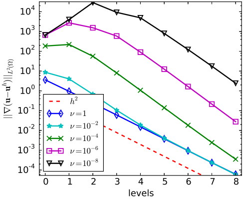
$$
\begin{aligned}
\boldsymbol{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix} = 200 \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix} \\
p &= \pi^2 (xy^3 \cos(2\pi x^2 y) - x^2 y \sin(2\pi xy)) + \frac{1}{8}
\end{aligned}
$$

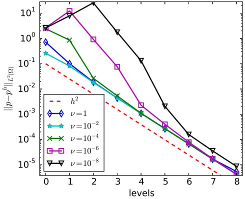




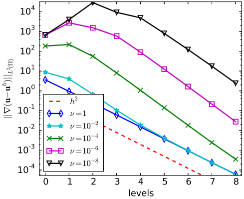
- $\boldsymbol{b} = \boldsymbol{u}$

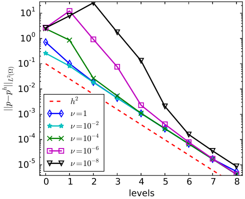
- Q_2/Q_1, convergence of errors for $c = 0$ and different values of ν



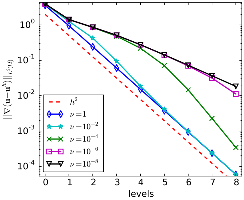


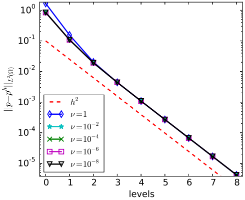
- Q_2/Q_1, convergence of errors for $c = 0$ and different values of ν



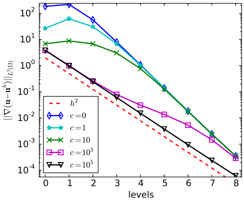


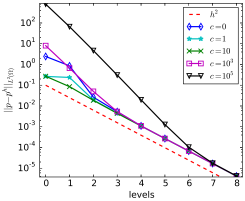
- Q_2/Q_1, convergence of errors for $c = 100$ and different values of ν



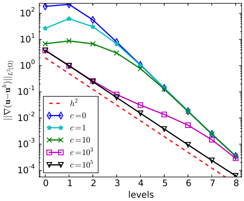


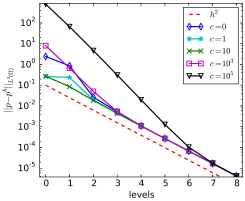
- Q_2/Q_1, convergence of errors for $\nu = 10^{-4}$ and different values of c





- Q_2/Q_1, convergence of errors for $\nu = 10^{-4}$ and different values of c





- summary
  - Galerkin discretization in some cases unstable

- principal idea
- given: linear partial differential equation in strong form

$$A_{\text{str}} u_{\text{str}} = f, \quad f \in L^2(\Omega)$$

- Galerkin discretization

$$a^h\left(u^h, v^h\right) = \left(f, v^h\right) \quad \forall\, v^h \in V^h$$

- needed: modification of strong operator $A^h_{\text{str}} \;:\; V^h \to L^2(\Omega)$
- residual

$$r^h\left(u^h\right) = A^h_{\text{str}} u^h - f \;\in L^2(\Omega)$$

- generally $r^h\left(u^h\right) \neq 0$

- principal idea (cont.)
- consider optimization problem

$$\arg\min_{u^h \in V^h} \left\| r^h\left(u^h\right)\right\|^2_{L^2(\Omega)} = \arg\min_{u^h \in V^h} \left(r^h\left(u^h\right), r^h\left(u^h\right)\right)$$

- necessary condition for solution (board p. 259)

$$\left(r^h\left(u^h\right), A^h_{\mathrm{str}} v^h\right) = 0$$

- principal idea (cont.)
- consider optimization problem

$$\operatorname*{arg\,min}_{u^h \in V^h} \left\| r^h\left(u^h\right) \right\|_{L^2(\Omega)}^2 = \operatorname*{arg\,min}_{u^h \in V^h} \left(r^h\left(u^h\right), r^h\left(u^h\right) \right)$$

- necessary condition for solution (board p. 259)

$$\left(r^h\left(u^h\right), A_{\mathrm{str}}^h v^h \right) = 0$$

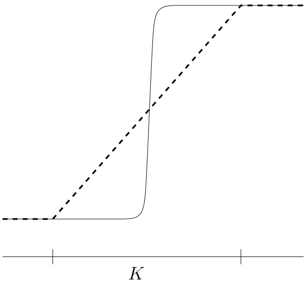
- generalization $\delta(\boldsymbol{x}) > 0$

$$\operatorname*{arg\,min}_{u^h \in V^h} \left\| \delta^{1/2} r^h\left(u^h\right) \right\|_{L^2(\Omega)}^2 = \operatorname*{arg\,min}_{u^h \in V^h} \left(\delta r^h\left(u^h\right), r^h\left(u^h\right) \right)$$

with necessary condition

$$\left(\delta r^h\left(u^h\right), A_{\mathrm{str}}^h v^h \right) = 0$$

- principal idea (cont.)
- minimizing residual alone: not good



- solid line – function with layer
- dashed line – optimal piecewise linear approximation

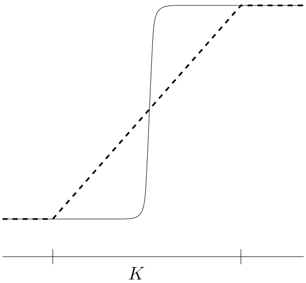
$\Longrightarrow$ residual large

- consider combination

$$a^h\left(u^h, v^h\right) + \left(\delta r^h\left(u^h\right), A^h_{\mathrm{str}} v^h\right) = \left(f, v^h\right) \quad \forall\, v^h \in V^h$$

optimal choice of weighting function $\delta(\boldsymbol{x})$ by numerical analysis

- principal idea (cont.)
- minimizing residual alone: not good



- solid line – function with layer
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- consider combination

$$a^h\left(u^h, v^h\right) + \left(\delta r^h\left(u^h\right), A^h_{\mathrm{str}} v^h\right) = \left(f, v^h\right) \quad \forall\, v^h \in V^h$$

optimal choice of weighting function $\delta(\boldsymbol{x})$ by numerical analysis

- example: Oseen equations, board p. 261

- SUPG/PSPG/grad-div stabilization
- find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$A_{\text{spg}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\text{spg}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \quad \forall \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h,$$

with $A_{\text{spg}} : \left(V \times \tilde{Q}\right) \times \left(V \times \tilde{Q}\right) \to \mathbb{R}$

$$\begin{aligned} & A_{\text{spg}}\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) \\ = \quad & \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) + \left((\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + c\boldsymbol{u}, \boldsymbol{v}\right) - \left(\nabla \cdot \boldsymbol{v}, p\right) + \left(\nabla \cdot \boldsymbol{u}, q\right) \\ & + \sum_{K \in \mathcal{T}^h} \mu_K \left(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}\right)_K + \sum_{E \in \mathcal{E}^h} \delta_E \left([|p|]_E, [|q|]_E\right)_E \\ & + \sum_{K \in \mathcal{T}^h} \left(-\nu \Delta \boldsymbol{u} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + c\boldsymbol{u} + \nabla p, \delta_K^v (\boldsymbol{b} \cdot \nabla)\boldsymbol{v} + \delta_K^p \nabla q\right)_K \end{aligned}$$

and $L_{\text{spg}} : \left(V \times \tilde{Q}\right) \to \mathbb{R}$

$$L_{\text{spg}}\left((\boldsymbol{v}, q)\right) = (\boldsymbol{f}, \boldsymbol{v}) + \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{f}, \delta_K^v (\boldsymbol{b} \cdot \nabla)\boldsymbol{v} + \delta_K^p \nabla q\right)_K$$

- SUPG/PSPG/grad-div stabilization (cont.)
- finite element error analysis in [1,2]
- $\delta_K = \delta_K^v = \delta_K^p$ for all $K \in \mathcal{T}^h$

$$\delta = \max_{K \in \mathcal{T}^h} \delta_K, \quad \mu = \max_{K \in \mathcal{T}^h} \mu_K$$

[1] Tobiska, Verfürth; SINUM 33, 107–127, 1996

[2] Roos, Stynes, Tobiska; *Robust numerical methods for singularly perturbed differential equations*, Springer, 2008

- SUPG/PSPG/grad-div stabilization (cont.)
- finite element error analysis in [1,2]
- $\delta_K = \delta_K^v = \delta_K^p$ for all $K \in \mathcal{T}^h$

$$\delta = \max_{K\in\mathcal{T}^h} \delta_K, \quad \mu = \max_{K\in\mathcal{T}^h} \mu_K$$

- no saddle point problem because of

$$\sum_{E\in\mathcal{E}^h} \delta_E \left([|p^h|]_E, [|q^h|]_E\right)_E + \sum_{K\in\mathcal{T}^h} \delta_K \left(\nabla p^h, \nabla q^h\right)_K$$

  - analysis for elliptic partial differential equations applicable
  - inf-sup stable spaces not necessary
  - choice of stabilization parameters affected by choice of finite element spaces

[1] Tobiska, Verfürth; SINUM 33, 107–127, 1996

[2] Roos, Stynes, Tobiska; *Robust numerical methods for singularly perturbed differential equations*, Springer, 2008

- properties
  - consistency

$$A_{\text{spg}}\left((\boldsymbol{u},p),\left(\boldsymbol{v}^h,q^h\right)\right)=L_{\text{spg}}\left(\left(\boldsymbol{v}^h,q^h\right)\right),\quad \forall\left(\boldsymbol{v}^h,q^h\right)\in V^h\times Q^h$$

  - Galerkin orthogonality

$$A_{\text{spg}}\left(\left(\boldsymbol{u}-\boldsymbol{u}^h,p-p^h\right),\left(\boldsymbol{v}^h,q^h\right)\right)=0,\quad \forall\left(\boldsymbol{v}^h,q^h\right)\in V^h\times Q^h$$

- mesh-dependent norm

$$\|(\boldsymbol{v},q)\|_{\mathrm{spg}} = \Big\{\nu\,\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\boldsymbol{v}\right\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{T}^h}\mu_K\,\|\nabla\cdot\boldsymbol{v}\|_{L^2(K)}^2$$
$$+ \sum_{E\in\mathcal{E}^h}\delta_E\,\|[|q|]_E\|_{L^2(E)}^2 + \sum_{K\in\mathcal{T}^h}\delta_K\,\|(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}+\nabla q\|_{L^2(K)}^2\Big\}^{1/2}$$

  - proof: similar to PSPG method
  - additional control on error of
    - divergence
    - pressure jumps
    - streamline derivative + gradient of pressure
  - norm with pressure: later

- existence and uniqueness of a solution
  - assumptions

$$\mu_K \geq 0, \quad 0 < \delta_K \leq \min\left\{\frac{h_K^2}{3\nu C_{\text{inv}}^2}, \frac{1}{3\left\|c\right\|_{L^\infty(K)}}\right\}$$

    $\delta_E > 0$ if $Q^h \not\subset C(\overline{\Omega})$
  - proof:
    - coercivity, $\forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$

$$A_{\text{spg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \geq \frac{1}{2}\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{spg}}^2$$

    - $\Longrightarrow$ system matrix non-singular

- stability

$$\left\|\left(\boldsymbol{u}^h, p^h\right)\right\|_{\mathrm{spg}}^2 \leq \frac{12}{5}\min\left\{\frac{\|\boldsymbol{f}\|_{H^{-1}(\Omega)}^2}{\nu}, \frac{\|\boldsymbol{f}\|_{L_2(\Omega)}^2}{c_0}\right\} + 4\sum_{K\in\mathcal{T}^h}\delta_K\|\boldsymbol{f}\|_{L^2(K)}^2$$

  - proof: as usual
  - estimate in stronger norm than for Galerkin finite element method
  - estimate for pressure with inf-sup condition possible

- norm for finite element error estimates

$$\|(\boldsymbol{v},q)\|_{\mathrm{spg,p}} = \left(\|(\boldsymbol{v},q)\|_{\mathrm{spg}} + w_{\mathrm{pres}}^{-2}\,\|q\|_{L^2(\Omega)}^2\right)^{1/2}$$

with

$$w_{\mathrm{pres}} = \max\left\{1, \nu^{-1/2}, \|c\|_{L^\infty(\Omega)}^{1/2}\right\}$$

for the interesting cases of small ν and large c: small contribution of the pressure

- norm for finite element error estimates

$$\|(\boldsymbol{v},q)\|_{\mathrm{spg,p}} = \left(\|(\boldsymbol{v},q)\|_{\mathrm{spg}} + w_{\mathrm{pres}}^{-2}\,\|q\|_{L^2(\Omega)}^2\right)^{1/2}$$

with

$$w_{\mathrm{pres}} = \max\left\{1, \nu^{-1/2}, \|c\|_{L^\infty(\Omega)}^{1/2}\right\}$$

for the interesting cases of small ν and large c: small contribution of the pressure

- first step: inf-sup conditions for A_{spg}

$$\inf_{\substack{(\boldsymbol{v}^h,q^h)\in V^h\times Q^h\\ \|(\boldsymbol{u}^h,p^h)\|_{\mathrm{spg,p}}=1}} \ \sup_{\substack{(\boldsymbol{w}^h,r^h)\in V^h\times Q^h\\ \|(\boldsymbol{v}^h,q^h)\|_{\mathrm{spg,p}}=1}} A_{\mathrm{spg}}\left(\left(\boldsymbol{v}^h,q^h\right),\left(\boldsymbol{w}^h,r^h\right)\right) \ge \beta_{\mathrm{spg}}$$

  - some conditions on stabilization parameters, e.g., $\delta_0 h_K^2 \le \delta_K$
  - proof very technical
  - $\beta_{\mathrm{spg}} = \mathcal{O}\left(\delta_0\right)$

- finite element error estimate

$$
\begin{aligned}
&\left\| \left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right) \right\|_{\mathrm{spg},p} \\
&\leq \quad C \Bigg[h^k \Bigg(\nu^{1/2} + \left(h + \delta^{1/2} h\right) \|c\|_{L^\infty(\Omega)}^{1/2} + \delta^{1/2} \|\boldsymbol{b}\|_{L^\infty(\Omega)}^{1/2} + \delta^{1/2} \\
&\qquad + \delta_0^{-1/2} + \gamma_0^{-1/2} + \mu^{1/2} \Bigg) \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} \\
&\qquad + h^l \Bigg(\delta^{1/2} + h \min\left\{ \nu^{-1/2}, \max_{K \in \mathcal{T}^h} \left\{ \mu_K^{-1/2} \right\} \right\} + h\omega_{\mathrm{pres}}^{-1} \\
&\qquad + \gamma^{1/2} \left(h + h^{1/2}\right) \Bigg) \|p\|_{H^{l+1}(\Omega)} \Bigg]
\end{aligned}
$$

  - $k \geq 1, l \geq 0$
  - C independent of the coefficients of the problem
  - proof: based on inf-sup condition A_{spg}

- optimal asymptotics for stabilization parameters, $\nu < h$, board p. 282
  - inf-sup stable discretizations with $k = l + 1$

$$\delta \sim h^2, \quad \mu \sim 1 \quad \Longrightarrow \quad \text{order of error reduction: } k$$

- optimal asymptotics for stabilization parameters, $\nu < h$, board p. 282
  - inf-sup stable discretizations with $k = l + 1$

$$\delta \sim h^2, \quad \mu \sim 1 \quad \Longrightarrow \quad \text{order of error reduction: } k$$

  - equal-order discretizations with $k = l \geq 1$

$$\delta \sim \mu \sim h \quad \Longrightarrow \quad \text{order of error reduction: } k + \frac{1}{2}$$

- optimal asymptotics for stabilization parameters, $\nu < h$, board p. 282
  - inf-sup stable discretizations with $k = l + 1$

$$\delta \sim h^2, \quad \mu \sim 1 \quad \Longrightarrow \quad \text{order of error reduction: } k$$

  - equal-order discretizations with $k = l \geq 1$

$$\delta \sim \mu \sim h \quad \Longrightarrow \quad \text{order of error reduction: } k + \frac{1}{2}$$

- optimal asymptotics for stabilization parameters, $\nu \geq h$
  - inf-sup stable discretizations with $k = l + 1$

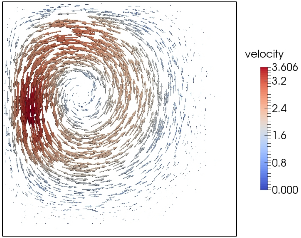
$$\delta \sim h^2, \quad \mu \sim 1 \quad \Longrightarrow \quad \text{order of convergence: } k$$

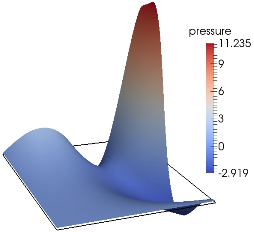
  - equal-order discretizations with $k = l \geq 1$

$$\delta \sim h^2, \quad \mu \sim 1 \quad \Longrightarrow \quad \text{order of convergence: } k$$

- analytical example which supports the error estimates
- prescribed solution

$$
\begin{aligned}
\boldsymbol{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix} = 200 \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix} \\
p &= \pi^2 (xy^3 \cos(2\pi x^2 y) - x^2 y \sin(2\pi x y)) + \frac{1}{8}
\end{aligned}
$$



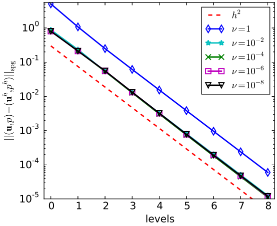


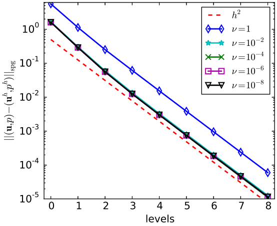
- $\boldsymbol{b} = \boldsymbol{u}$

- Q_2/Q_1 finite element
- stabilization parameters (based on numerical simulations from [1])

$$\mu_K = 0.2, \quad \delta_K = 0.1 h_K^2$$

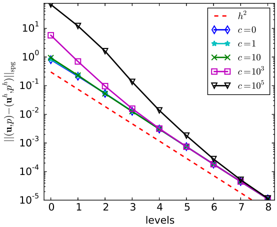
- convergence of errors for $c = 0$ and $c = 100$, different values of ν





[1] Matthies, Lube, Röhe, Comput. Methods Appl. Math. 9, 368 – 390, 2009

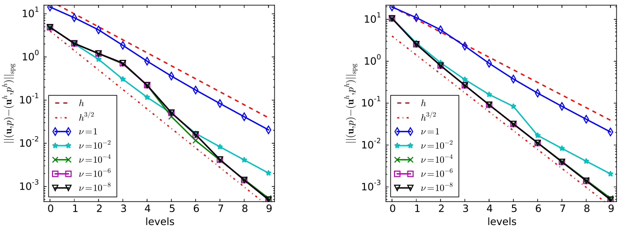
- Q_2/Q_1, convergence of errors for $\nu = 10^{-4}$ and different values of c



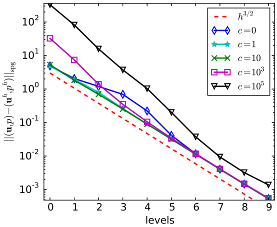
- P_1/P_1 finite element
- stabilization parameters

$$\delta_K = \begin{cases} 0.5h_K & \text{if } \nu < h_K, \\ 0.5h_K^2 & \text{else,} \end{cases} \qquad \mu_K = 0.5h_K$$

- convergence of errors for $c = 0$ and $c = 100$, different values of ν



- P_1/P_1, convergence of errors for $\nu = 10^{-4}$ and different values of c



- implementation: same approach as for Stokes equations
- grad-div term leads to matrix block

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix} \quad \text{instead of} \quad \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

- PSPG term introduces pressure-pressure couplings
- SUPG term influences velocity-velocity coupling and the pressure (ansatz) - velocity (test) coupling
- final system

$$\begin{pmatrix} A & D \\ B & -C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{f_p} \end{pmatrix}$$

much more matrix blocks to store than for Galerkin FEM

- summary and remarks
  - errors $\left\|(\boldsymbol{u}, p) - (\boldsymbol{u}^h, p^h)\right\|_{\mathrm{spg,p}}$ independent of ν
  - versions without pressure couplings available
    - only for inf-sup stable pairs of finite elements
    - easier to implement than SUPG/PSPG/grad-div stabilization
  - numerical analysis in [1,2,3]
- other stabilizations proposed in the literature

[1] Tobiska, Verfürth, SINUM 33, 107–127, 1996

[2] Lube, Rapin, M3AS 16, 949–966, 2006

[3] Matthies, Lube, Röhe, Comput. Methods Appl. Math. 9, 368–390, 2009

7. The Stationary Navier–Stokes Equation

- continuous equation

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

  for simplicity: homogeneous Dirichlet boundary conditions
- difficulties:
  - coupling of velocity and pressure
  - dominating convection
  - nonlinear

- different forms of the convective term

$$
\begin{array}{lcl}
(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} & : & \text{convective form}, \\
\nabla\cdot\left(\boldsymbol{u}\boldsymbol{u}^T\right) & : & \text{divergence form}, \\
(\nabla\times\boldsymbol{u})\times\boldsymbol{u} & : & \text{rotational form}
\end{array}
$$

  - convective form and divergence form equivalent if $\nabla\cdot\boldsymbol{u}=0$ (apply product rule to divergence form)
  - convective form and rotational form

$$(\nabla\times\boldsymbol{u})\times\boldsymbol{u}+\frac{1}{2}\nabla\left(\boldsymbol{u}^T\boldsymbol{u}\right)=(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$$

definition of new pressure (Bernoulli pressure) in rotational form

$$p_{\text{Bern}}=p+\frac{1}{2}\boldsymbol{u}^T\boldsymbol{u}$$

- different forms of the convective term (cont.)
  - recent proposal [1]: EMAC (energy momentum and angular momentum conserving)

$$2\mathbb{D}\left(\boldsymbol{u}\right)\boldsymbol{u}+\left(\nabla\cdot\boldsymbol{u}\right)\boldsymbol{u}$$

    with new pressure (negative of Bernoulli pressure)
  - derivation based on conservation of
    - kinetic energy ($\nu = 0, \boldsymbol{f} = \boldsymbol{0}$)
    - linear momentum ($\boldsymbol{f}$ with vanishing linear momentum)
    - angular momentum ($\boldsymbol{f}$ with vanishing angular momentum)
    - helicity ($\nu = 0$)
    - 2d enstrophy ($\nu = 0$)
    - vorticity ($\nu = 0$)
  - none of the other forms preserves all these quantities
  - first numerical experience: often among best disc. of nonlinear term

[1] Charnyi, Heister, Olshanskii, Rebholz; J. Comput. Phys. 337, 289–308, 2017

- variational form of the steady-state Navier–Stokes equations: Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} (\nu\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, p) &= (\boldsymbol{f}, \boldsymbol{v}), \\ -(\nabla\cdot\boldsymbol{u}, q) &= 0 \end{aligned}$$

  for all $(\boldsymbol{v}, q) \in V \times Q$
- equivalent: Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$(\nu\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, p) + (\nabla\cdot\boldsymbol{u}, q) = (\boldsymbol{f}, \boldsymbol{v})$$

  for all $(\boldsymbol{v}, q) \in V \times Q$

- properties of convective term
  - linear in each component (trilinear)
  - $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H^1(\Omega)$, product rule

$$((\boldsymbol{u} \cdot \nabla)\, \boldsymbol{v}, \boldsymbol{w}) = \left(\nabla \cdot \left(\boldsymbol{v}\boldsymbol{u}^T\right), \boldsymbol{w}\right) - ((\nabla \cdot \boldsymbol{u})\, \boldsymbol{v}, \boldsymbol{w})$$

  - $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H^1(\Omega)$, product rule

$$((\boldsymbol{u} \cdot \nabla)\, \boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{u}, \nabla\, (\boldsymbol{v} \cdot \boldsymbol{w})) - ((\boldsymbol{u} \cdot \nabla)\, \boldsymbol{w}, \boldsymbol{v})$$

- convective terms in the variational formulation
  - convective form

$$n_{\text{conv}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})$$

  - divergence form

$$n_{\text{div}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = n_{\text{conv}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) + \frac{1}{2}\left((\nabla\cdot\boldsymbol{u})\,\boldsymbol{v},\boldsymbol{w}\right)$$

  - rotational form

$$n_{\text{rot}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = ((\nabla\times\boldsymbol{u})\times\boldsymbol{v},\boldsymbol{w})$$

    with momentum equation

$$(\nu\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + n_{\text{rot}}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - (\nabla\cdot\boldsymbol{v},p_{\text{Bern}}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall\,\boldsymbol{v}\in V$$

  - skew-symmetric form (for $\boldsymbol{u}$ weakly divergence-free, $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on Γ)

$$n_{\text{skew}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \frac{1}{2}\left(n_{\text{conv}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) - n_{\text{conv}}(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})\right)$$

- further properties of convective term
- vanishing
  - rotational, skew-symmetric, and divergence form

$$n_{\text{rot}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = n_{\text{skew}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = n_{\text{div}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0$$

  - convective: if $\boldsymbol{u}$ weakly divergence-free and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ

$$n_{\text{conv}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0$$

- further properties of convective term
- vanishing
  - rotational, skew-symmetric, and divergence form

$$n_{\text{rot}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = n_{\text{skew}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = n_{\text{div}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0$$

  - convective: if $\boldsymbol{u}$ weakly divergence-free and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ

$$n_{\text{conv}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0$$

- estimates: $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in H^1(\Omega)$

$$\begin{aligned} |n_{\text{conv}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| &\le C \|\boldsymbol{u}\|_{H^1(\Omega)} \|\nabla\boldsymbol{v}\|_{L^2(\Omega)} \|\boldsymbol{w}\|_{H^1(\Omega)}, \\ |n_{\text{skew}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| &\le C \|\boldsymbol{u}\|_{H^1(\Omega)} \|\boldsymbol{v}\|_{H^1(\Omega)} \|\boldsymbol{w}\|_{H^1(\Omega)} \end{aligned}$$

  - proof: board, p. 309

- existence and uniqueness of a solution
  - $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, bounded domain with Lipschitz boundary
  - $\boldsymbol{f} \in H^{-1}(\Omega)$
  - then: existence
- main ideas of the proof
  - equivalent problem in the divergence-free subspace, only velocity
  - consider problem in finite-dimensional spaces (Galerkin method)
  - fixed point equation, existence of a solution of the finite-dimensional problems: fixed point theorem of Brouwer
  - dimension of the spaces $\to \infty$: show subsequence of the solutions tends to a solution of the problem in the divergence-free subspace
  - existence of the pressure: inf-sup condition

- existence and uniqueness of a solution (cont.)
  - ν sufficiently large, i.e.,

$$\|\boldsymbol{f}\|_{H^{-1}(\Omega)} \sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V} \frac{((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})}{\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)}} < \nu^2$$

  - then: uniqueness
- main idea of the proof
  - construct a contraction, apply Banach's fixed point theorem
  - use result of existence and uniqueness of solution for Oseen equations

- existence and uniqueness of a solution (cont.)
  - ν sufficiently large, i.e.,

$$\|\boldsymbol{f}\|_{H^{-1}(\Omega)} \sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V} \frac{((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})}{\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}\,\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}\,\|\nabla\boldsymbol{w}\|_{L^2(\Omega)}} < \nu^2$$

  - then: uniqueness
- main idea of the proof
  - construct a contraction, apply Banach's fixed point theorem
  - use result of existence and uniqueness of solution for Oseen equations
- numerical simulations
  - case of unique solution is of interest
  - steady-state solutions unstable in non-unique case, solve time-dependent problem

- stability

$$
\begin{aligned}
\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} &\le \frac{1}{\nu}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}, \\
\|p\|_{L^2(\Omega)} &\le \frac{1}{\beta_{\mathrm{is}}}\left(2\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + \frac{C}{\nu^2}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)}\right)
\end{aligned}
$$

  - proof: as usual, using

$$
n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}) = 0
$$

- Galerkin finite element method

$$
\begin{array}{rcll}
\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) + n\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) & = & \left(\boldsymbol{f}, \boldsymbol{v}^h\right) & \forall\, \boldsymbol{v}^h \in V^h, \\
-\left(\nabla \cdot \boldsymbol{u}^h, q^h\right) & = & 0 & \forall\, q^h \in Q^h
\end{array}
$$

- inf-sup stable pair of finite element spaces

- Galerkin finite element method

$$
\begin{array}{rcll}
\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) + n\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) & = & \left(\boldsymbol{f}, \boldsymbol{v}^h\right) & \forall\, \boldsymbol{v}^h \in V^h, \\
-\left(\nabla \cdot \boldsymbol{u}^h, q^h\right) & = & 0 & \forall\, q^h \in Q^h
\end{array}
$$

- inf-sup stable pair of finite element spaces
- finite element error analysis for $n_{\mathrm{skew}}(\cdot,\cdot,\cdot)$

$$
n_{\mathrm{skew}}\left(\boldsymbol{u}^h, \boldsymbol{v}^h, \boldsymbol{v}^h\right) = \frac{1}{2}\left(n_{\mathrm{conv}}\left(\boldsymbol{u}^h, \boldsymbol{v}^h, \boldsymbol{v}^h\right) - n_{\mathrm{conv}}\left(\boldsymbol{u}^h, \boldsymbol{v}^h, \boldsymbol{v}^h\right)\right) = 0
$$

note that in general $\boldsymbol{u}^h \notin V_{\mathrm{div}} \quad \Longrightarrow$

$$
n_{\mathrm{conv}}\left(\boldsymbol{u}^h, \boldsymbol{v}^h, \boldsymbol{v}^h\right) \neq 0
$$

- Galerkin finite element method

$$
\begin{array}{rcll}
\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) + n\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) & = & \left(\boldsymbol{f}, \boldsymbol{v}^h\right) & \forall\, \boldsymbol{v}^h \in V^h, \\
-\left(\nabla \cdot \boldsymbol{u}^h, q^h\right) & = & 0 & \forall\, q^h \in Q^h
\end{array}
$$

- inf-sup stable pair of finite element spaces
- finite element error analysis for $n_{\mathrm{skew}}(\cdot,\cdot,\cdot)$

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$$

note that in general $\boldsymbol{u}^h \notin V_{\mathrm{div}} \quad \Longrightarrow$

$$
n_{\mathrm{conv}}\left(\boldsymbol{u}^h, \boldsymbol{v}^h, \boldsymbol{v}^h\right) \neq 0
$$

- same as for continuous problem:
  - existence, uniqueness
  - stability

- Finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain with polyhedral boundary
  - $\nu^{-2} \|\boldsymbol{f}\|_{H^{-1}(\Omega)}$ be sufficiently small such that unique solution
  - inf-sup stable finite element spaces $V^h \times Q^h$

$$
\begin{aligned}
&\left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)} \\
&\quad \le C\left(\left(1 + \frac{1}{\nu^2}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\right)\left(1 + \frac{1}{\beta_{\mathrm{is}}^h}\right)\inf_{\boldsymbol{v}^h \in V^h}\left\|\nabla\left(\boldsymbol{u} - \boldsymbol{v}^h\right)\right\|_{L^2(\Omega)} \right.\\
&\qquad \left. + \frac{1}{\nu}\inf_{q^h \in Q^h}\left\|p - q^h\right\|_{L^2(\Omega)}\right)
\end{aligned}
$$

  - proof: main ideas and treatment of nonlinear term: board, p. 320

- Finite element error estimate for the $L^2(\Omega)$ norm of the pressure

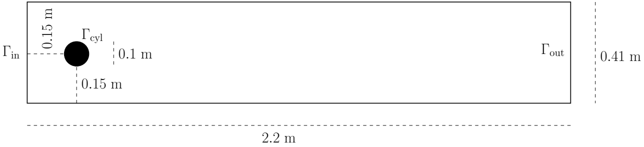
$$
\begin{aligned}
&\left\|p - p^h\right\|_{L^2(\Omega)} \\
&\quad \le \quad C\frac{\nu}{\beta_{\mathrm{is}}^h}\left(\left(1 + \frac{1}{\nu^2}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\right)^2\left(1 + \frac{1}{\beta_{\mathrm{is}}^h}\right)\inf_{\boldsymbol{v}^h\in V^h}\left\|\nabla\left(\boldsymbol{u} - \boldsymbol{v}^h\right)\right\|_{L^2(\Omega)}\right. \\
&\qquad\qquad \left. + C\frac{\nu}{\beta_{\mathrm{is}}^h}\left(1 + \frac{1}{\nu^2}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\right)\inf_{q^h\in Q^h}\left\|p - q^h\right\|_{L^2(\Omega)}\right)
\end{aligned}
$$

- Finite element error estimate for the $L^2(\Omega)$ norm of the pressure

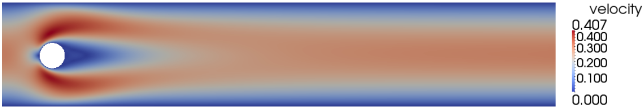
$$
\begin{aligned}
&\left\|p - p^h\right\|_{L^2(\Omega)} \\
\le\quad & C\frac{\nu}{\beta_{\text{is}}^h}\left(\left(1+\frac{1}{\nu^2}\left\|\boldsymbol{f}\right\|_{H^{-1}(\Omega)}\right)^2\left(1+\frac{1}{\beta_{\text{is}}^h}\right)\inf_{\boldsymbol{v}^h\in V^h}\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(\Omega)}\right. \\
&\left.+C\frac{\nu}{\beta_{\text{is}}^h}\left(1+\frac{1}{\nu^2}\left\|\boldsymbol{f}\right\|_{H^{-1}(\Omega)}\right)\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\right)
\end{aligned}
$$

- analytical results can be supported numerically by analytical test examples

- Example: steady-state flow around a cylinder at Re $= 20$
  - domain

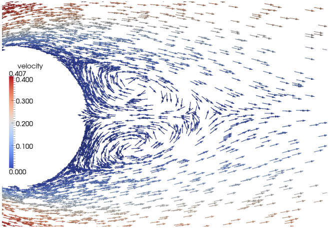


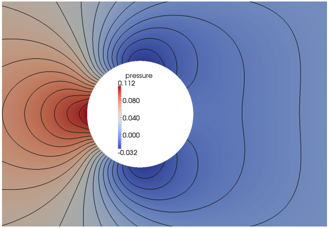
  - velocity



  - pressure

- Example: steady-state flow around a cylinder at Re $= 20$
  - at the cylinder





- important: drag and lift coefficient at the cylinder

$$
\begin{aligned}
c_{\mathrm{drag}} &= \frac{2}{\rho d U_{\mathrm{mean}}^2} \int_{\Gamma_{\mathrm{cyl}}} \left(\mu \frac{\partial \mathbf{v}_{\boldsymbol{t}}}{\partial \boldsymbol{n}} n_y - P n_x \right) \, ds, \\
c_{\mathrm{lift}} &= -\frac{2}{\rho d U_{\mathrm{mean}}^2} \int_{\Gamma_{\mathrm{cyl}}} \left(\mu \frac{\partial \boldsymbol{v}_{\boldsymbol{t}}}{\partial \boldsymbol{n}} n_x + P n_y \right) \, ds
\end{aligned}
$$

- important: drag and lift coefficient at the cylinder

$$
\begin{aligned}
c_{\text{drag}} &= \frac{2}{\rho d U_{\text{mean}}^2} \int_{\Gamma_{\text{cyl}}} \left(\mu \frac{\partial \mathbf{v}_{\boldsymbol{t}}}{\partial \boldsymbol{n}} n_y - P n_x \right) \, ds, \\
c_{\text{lift}} &= -\frac{2}{\rho d U_{\text{mean}}^2} \int_{\Gamma_{\text{cyl}}} \left(\mu \frac{\partial \boldsymbol{v}_{\boldsymbol{t}}}{\partial \boldsymbol{n}} n_x + P n_y \right) \, ds
\end{aligned}
$$

- reformulation with volume integrals possible, long but elementary derivation, e.g.,

$$
c_{\text{drag}} = -\frac{2U^2}{dU_{\text{mean}}^2} \left((\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{w}_d) + n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}_d) - (\nabla \cdot \boldsymbol{w}_d, p) - (\boldsymbol{f}, \boldsymbol{w}_d) \right)
$$

for any function $\boldsymbol{w}_d \in H^1(\Omega)$ with $\boldsymbol{w}_d = \mathbf{0}$ on $\Gamma \setminus \Gamma_{\text{cyl}}$ and $\boldsymbol{w}_d|_{\Gamma_{\text{cyl}}} = (1,0)^T$

- reference values
  - [1] : compiled from simulations of different groups

$$c_{\mathrm{drag,ref}} \in [5.57, 5.59], \quad c_{\mathrm{lift,ref}} \in [0.104, 0.110]$$

  - [2] : do-nothing conditions at outlet

$$c_{\mathrm{drag,ref}} = 5.57953523384, \quad c_{\mathrm{lift,ref}} = 0.010618948146$$

  - [3] : Dirichlet conditions at outlet

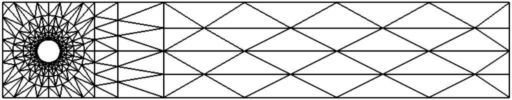
$$c_{\mathrm{drag,ref}} = 5.57953523384, \quad c_{\mathrm{lift,ref}} = 0.010618937712$$

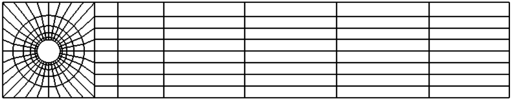
[1] Schäfer, Turek; Notes on Numerical Fluid Mechanics 52, 547–566, 1996

[2] Nabh; PhD thesis, Heidelberg, 1998

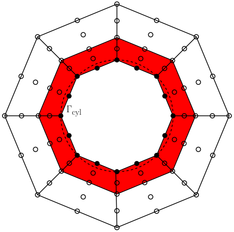
[3] J., Matthies; IJNMF 37, 885–903, 2001

- initial grids

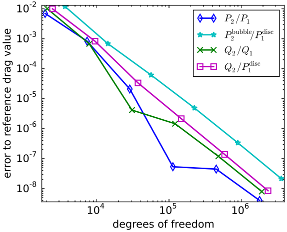


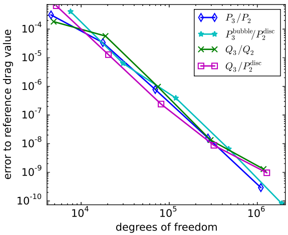


- patch for test function in computation of coefficients, Q_2

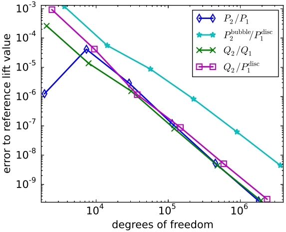


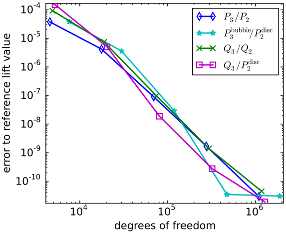
- convective form of convective term
- do-nothing boundary conditions
- convergence of drag coefficient





- convergence of lift coefficient





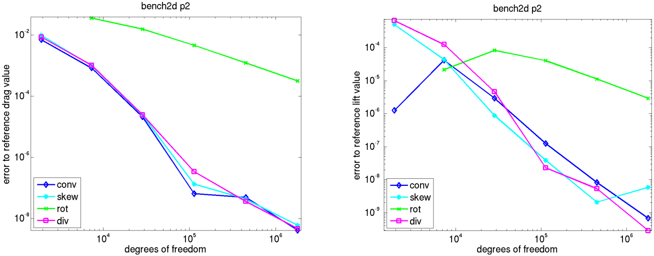
- different forms of the convective term
  - structure of matrix for convective, divergence, and skew-symmetric form

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

  - structure of matrix for rotational form

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

- different forms of the convective term, P_2/P_1



- rotational form
  - reconstructed pressure has boundary layers, inaccurate results

- schemes for solving the nonlinearity
- fixed point iteration

$$\begin{pmatrix} \boldsymbol{u}^{(m+1)} \\ p^{(m+1)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}^{(m)} \\ p^{(m)} \end{pmatrix} - \vartheta \boldsymbol{N}_{\text{lin}}^{-1} \left(\begin{pmatrix} (\boldsymbol{f}, \boldsymbol{v}) \\ 0 \end{pmatrix} - \boldsymbol{N}\left(\boldsymbol{u}^{(m)}; \boldsymbol{u}^{(m)}, p^{(m)} \right) \right)$$

with

$$\boldsymbol{N}(\boldsymbol{w}; \boldsymbol{u}, p) = \begin{pmatrix} a(\boldsymbol{u}, \boldsymbol{v}) + n(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) \\ b(\boldsymbol{u}, q) \end{pmatrix}$$

$\boldsymbol{N}_{\text{lin}}$ – linear operator
$\vartheta \in (0, 1]$ – damping factor

- fixed point iteration
  - linear system to be solved

$$\boldsymbol{N}_{\text{lin}} \begin{pmatrix} \delta \boldsymbol{u}^{(m+1)} \\ \delta p^{(m+1)} \end{pmatrix} = \left(\begin{pmatrix} (\boldsymbol{f}, \boldsymbol{v}) \\ 0 \end{pmatrix} - \boldsymbol{N} \left(\boldsymbol{u}^{(m)}; \boldsymbol{u}^{(m)}, p^{(m)} \right) \right)$$

  - setting

$$\begin{pmatrix} \delta \boldsymbol{u}^{(m+1)} \\ \delta p^{(m+1)} \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{u}}^{(m+1)} - \boldsymbol{u}^{(m)} \\ \tilde{p}^{(m+1)} - p^{(m)} \end{pmatrix},$$

then

$$\boldsymbol{N}_{\text{lin}} \begin{pmatrix} \tilde{\boldsymbol{u}}^{(m+1)} \\ \tilde{p}^{(m+1)} \end{pmatrix} = \left(\begin{pmatrix} (\boldsymbol{f}, \boldsymbol{v}) \\ 0 \end{pmatrix} - \boldsymbol{N} \left(\boldsymbol{u}^{(m)}; \boldsymbol{u}^{(m)}, p^{(m)} \right) \right) + \boldsymbol{N}_{\text{lin}} \begin{pmatrix} \boldsymbol{u}^{(m)} \\ p^{(m)} \end{pmatrix}$$

- iteration with Stokes equations

$$\boldsymbol{N}_{\text{lin}} = \boldsymbol{N}\left(\boldsymbol{0}; \tilde{\boldsymbol{u}}^{(m+1)}, \tilde{p}^{(m+1)}\right)$$

- then

$$\begin{aligned}
&\begin{pmatrix} a(\tilde{\boldsymbol{u}}^{(m+1)}, \boldsymbol{v}) + b(\boldsymbol{v}, \tilde{p}^{(m+1)}) \\ b(\tilde{\boldsymbol{u}}^{(m+1)}, q) \end{pmatrix} \\
&= \quad \begin{pmatrix} (\boldsymbol{f}, \boldsymbol{v}) - a(\boldsymbol{u}^{(m)}, \boldsymbol{v}) - n(\boldsymbol{u}^{(m)}, \boldsymbol{u}^{(m)}, \boldsymbol{v}) - b(\boldsymbol{v}, \tilde{p}^{(m)}) \\ -b(\tilde{\boldsymbol{u}}^{(m)}, q) \end{pmatrix} \\
&\qquad + \begin{pmatrix} a(\boldsymbol{u}^{(m)}, \boldsymbol{v}) + b(\boldsymbol{v}, \tilde{p}^{(m)}) \\ b(\tilde{\boldsymbol{u}}^{(m)}, q) \end{pmatrix} = \begin{pmatrix} (\boldsymbol{f}, \boldsymbol{v}) - n(\boldsymbol{u}^{(m)}, \boldsymbol{u}^{(m)}, \boldsymbol{v}) \\ 0 \end{pmatrix}
\end{aligned}$$

  - converges only if ν is sufficiently large
  - not recommended

- iteration with Oseen-type equations, Picard iteration

$$\boldsymbol{N}_{\mathrm{lin}} = \boldsymbol{N}\left(\boldsymbol{u}^{(m)}; \tilde{\boldsymbol{u}}^{(m+1)}, \tilde{p}^{(m+1)}\right)$$

- then

$$\begin{aligned}
&\begin{pmatrix} a(\tilde{\boldsymbol{u}}^{(m+1)}, \boldsymbol{v}) + n(\boldsymbol{u}^{(m)}, \tilde{\boldsymbol{u}}^{(m+1)}, \boldsymbol{v}) + b(\boldsymbol{v}, \tilde{p}^{(m+1)}) \\ b(\tilde{\boldsymbol{u}}^{(m+1)}, q) \end{pmatrix} \\
&= \quad \begin{pmatrix} (\boldsymbol{f}, \boldsymbol{v}) - a(\boldsymbol{u}^{(m)}, \boldsymbol{v}) - n(\boldsymbol{u}^{(m)}, \boldsymbol{u}^{(m)}, \boldsymbol{v}) - b(\boldsymbol{v}, \tilde{p}^{(m)}) \\ -b(\boldsymbol{u}^{(m)}, q) \end{pmatrix} \\
&\qquad + \begin{pmatrix} a(\boldsymbol{u}^{(m)}, \boldsymbol{v}) + n(\boldsymbol{u}^{(m)}, \boldsymbol{u}^{(m)}, \boldsymbol{v}) + b(\boldsymbol{v}, \tilde{p}^{(m)}) \\ b(\boldsymbol{u}^{(m)}, q) \end{pmatrix} \\
&= \quad \begin{pmatrix} (\boldsymbol{f}, \boldsymbol{v}) \\ 0 \end{pmatrix}
\end{aligned}$$

- widely used

- Newton's method
- linear operator is derivative of the nonlinear operator at the current position

$$\boldsymbol{N}_{\text{lin}} = D\boldsymbol{N}\begin{pmatrix}\boldsymbol{u}^{(m)}\\ p^{(m)}\end{pmatrix}$$

  - with Gâteaux derivative at $(\boldsymbol{u}, p)^T$

$$\begin{aligned} D\boldsymbol{N}\begin{pmatrix}\boldsymbol{u}\\ p\end{pmatrix} &= \lim_{\varepsilon\to 0}\frac{\boldsymbol{N}(\boldsymbol{u}+\varepsilon\boldsymbol{\phi};\boldsymbol{u}+\varepsilon\boldsymbol{\phi}, p+\varepsilon\psi) - \boldsymbol{N}(\boldsymbol{u};\boldsymbol{u},p)}{\varepsilon} \\ &= \boldsymbol{N}(\boldsymbol{\phi};\boldsymbol{u},p) + \boldsymbol{N}(\boldsymbol{u};\boldsymbol{\phi},p) + \boldsymbol{N}(\boldsymbol{u},\boldsymbol{u},\psi)\end{aligned}$$

  - inserting and collecting terms

$$\begin{pmatrix} a(\tilde{\boldsymbol{u}}^{(m+1)},\boldsymbol{v}) + n(\boldsymbol{u}^{(m)},\tilde{\boldsymbol{u}}^{(m+1)},\boldsymbol{v}) + n(\tilde{\boldsymbol{u}}^{(m+1)},\boldsymbol{u}^{(m)},\boldsymbol{v}) + b(\boldsymbol{v},\tilde{p}^{(m+1)}) \\ b(\tilde{\boldsymbol{u}}^{(m+1)}, q)\end{pmatrix}$$

$$= \begin{pmatrix}(\boldsymbol{f},\boldsymbol{v}) + n(\boldsymbol{u}^{(m)},\boldsymbol{u}^{(m)},\boldsymbol{v}) \\ 0\end{pmatrix}$$

  - analytical properties of term $n(\tilde{\boldsymbol{u}}^{(m+1)},\boldsymbol{u}^{(m)},\boldsymbol{v})$ unclear

- implementation
  - same principal approach as for Stokes and Oseen equations
  - inf-sup stable finite elements lead to linear saddle point problems in fixed point iteration

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

- implementation
  - same principal approach as for Stokes and Oseen equations
  - inf-sup stable finite elements lead to linear saddle point problems in fixed point iteration

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

  - convective form of convective term
    - Picard iteration

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

    - Newton iteration

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

- example: Picard vs. Newton
  - analytical solution
  - Q_2/P_1^{disc}
  - exact solution of linear systems vs inexact solution
    - inexact: reduce Euclidean norm of residual by factor 10, at most 10 iterations

- number of iterations for solving the nonlinear problem
  - 'not conv.': solution was not obtained within 100 iterations

		Picard iteration		Newton's method	
ν	level/lin. solver	inexact	exact	inexact	exact
$1/100$	2	14	26	8	5
	3	15	14	7	5
	4	14	14	7	5
	5	13	13	7	5
	6	13	13	7	5
$1/500$	2	39	not conv.	not conv.	not conv.
	3	32	32	not conv.	not conv.
	4	30	29	35	8
	5	29	28	52	8
	6	28	27	not conv.	8
$1/1000$	2	not conv.	not conv.	not conv.	not conv.
	3	36	57	not conv.	not conv.
	4	35	33	not conv.	not conv.
	5	35	31	not conv.	not conv.
	6	33	30	not conv.	not conv.

- Picard method
  - larger convergence radius
  - more robust
- Newton's method
  - faster in case of convergence and exact solution of linear systems
  - properties of term $n(\tilde{\boldsymbol{u}}^{(m+1)}, \boldsymbol{u}^{(m)}, \boldsymbol{v})$ not clear

- residual-based (and other) stabilizations possible
  - better: solve time-dependent problem

- discretization and linearization leads to algebraic linear system of equations

$$\mathscr{A}\underline{x} = \begin{pmatrix} A & D \\ B & -C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{f_p} \end{pmatrix} = \underline{y},$$

with

$$A \in \mathbb{R}^{dN_v \times dN_v},\ D \in \mathbb{R}^{dN_v \times N_p},\ B \in \mathbb{R}^{N_p \times dN_v},\ C \in \mathbb{R}^{N_p \times N_p},$$
$$\underline{u}, \underline{f} \in \mathbb{R}^{dN_v},\ \underline{p}, \underline{f_p} \in \mathbb{R}^{N_p},$$

such that

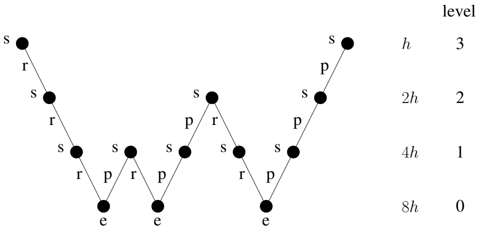
$$\mathscr{A} \in \mathbb{R}^{(dN_v+N_p)\times(dN_v+N_p)}, \quad \underline{x}, \underline{y} \in \mathbb{R}^{dN_v+N_p}$$

- sparse matrices
- efficiency of simulations depends strongly on efficiency of solution of these systems

- sparse direct solvers
  - MUMPS, pardiso, UMFPACK
  - black box, easy to use
  - improved considerably in the last 20 years
- iterative solvers
  - (flexible) GMRES(restart), BiCGStab, . . .
  - generally not necessary to solve linear systems very accurately
    - common strategy: reduce Euclidean norm of residual vector by factor 10
  - need preconditioner

- multigrid preconditioner
  - needs hierarchy of grids
  - grid transfer operators (restriction, prolongation)
  - smoother (iterative method for damping high frequency error components)
  - coarse grid solver, e.g., direct solver

F-cycle



- multigrid preconditioner (cont.)
  - smoother essentially for efficiency
    - for saddle point problems only block Gauss–Seidel smoothers efficient (Vanka smoothers) [1], multiplicative Vanka smoother
    - coupled treatment of velocity and pressure
  - multigrid methods take time for implementation
  - need some effort for parallelization

[1] Vanka; J. Comput. Phys. 65, 138–158, 1986

- decomposition of velocity d.o.f. $\mathcal{V}_h$ and pressure d.o.f. $\mathcal{Q}_h$

$$\mathcal{V}_h = \cup_{j=1}^{J} \mathcal{V}_{hj}, \quad \mathcal{Q}_h = \cup_{j=1}^{J} \mathcal{Q}_{hj}$$

- $\mathcal{A}_j$ matrix block $\mathcal{A}$ which is connected to $\mathcal{W}_{hj} = \mathcal{V}_{hj} \cup \mathcal{Q}_{hj}$

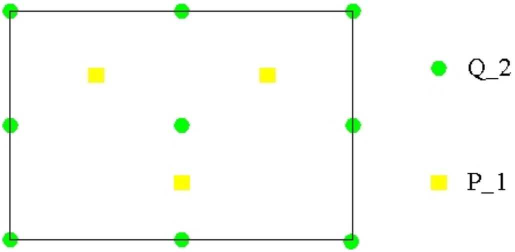
$$\mathcal{A}_j = \begin{pmatrix} A_j & B_j \\ C_j & 0 \end{pmatrix} \in \mathbb{R}^{\dim(\mathcal{W}_{hj}) \times \dim(\mathcal{W}_{hj})}$$

- one application of multiplicative Vanka smoother: for $j = 1, \ldots, J$

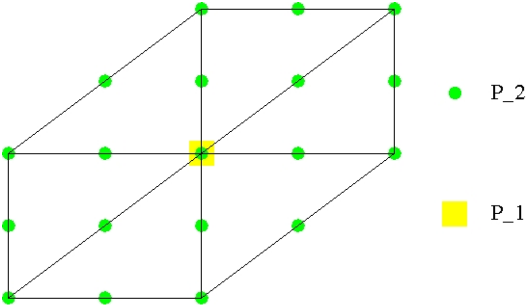
$$\begin{pmatrix} u \\ p \end{pmatrix}_j := \begin{pmatrix} u \\ p \end{pmatrix}_j + \mathcal{A}_j^{-1} \left(\begin{pmatrix} f \\ g \end{pmatrix} - \mathcal{A} \begin{pmatrix} u \\ p \end{pmatrix} \right)_j$$

- strategy:
  - choose $\mathcal{Q}_{hj}$
  - $\mathcal{V}_{hj}$ all velocity d.o.f. which are connected to pressure d.o.f. in $\mathcal{Q}_{hj}$

- discontinuous pressure approximation
- $\mathcal{W}_{hj}$: all d.o.f. which are connected to one mesh cell
- J : number of mesh cells



- continuous pressure approximation
- $\dim Q_{hj} = 1$ for all j
- J : number of pressure d.o.f.

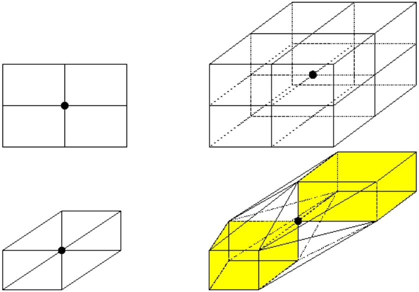


- mesh cell oriented Vanka smoother

	2d			3d		
	velo	pressure	total	velo	pressure	total
Q_1^{nc}/Q_0 (R/T)	4	1	9	6	1	19
Q_2/P_1^{disc}	9	3	21	27	4	85
Q_3/P_2^{disc}	16	6	38	64	10	202
P_1^{nc}/P_0 (C/R)	3	1	7	4	1	13

- same size for all mesh cells

- pressure node oriented Vanka smoother



	2d			3d		
	velo	pressure	total	velo	pressure	total
Q_2/Q_1	25	1	51	125	1	376
Q_3/Q_2	49	1	99	343	1	1030
P_2/P_1	19	1	39	65	1	196
P_3/P_2	37	1	75	175	1	526

- smoothers that treat velocity and pressure in decoupled way
  - Least Squares Commutator (LSC) preconditioner [1,2]
  - augmented Lagrangian-based preconditioner [3]

[1] Elman, Howle, Shadid, Shuttleworth, Tuminaro; SIAM J. Sci. Comput. 27, 1651–1668, 2006

[2] Elman, Silvester, Wathen; Oxford University Press, 2014

[3] Benzi, Wang; SIAM J. Sci. Comput. 33, 2761–2784, 2011

- smoothers that treat velocity and pressure in decoupled way
  - Least Squares Commutator (LSC) preconditioner [1,2]
  - augmented Lagrangian-based preconditioner [3]
- LSC preconditioner
  - starting point

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ BA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B^T \\ 0 & S \end{pmatrix}$$

with Schur complement

$$S = -BA^{-1}B^T$$

[1] Elman, Howle, Shadid, Shuttleworth, Tuminaro; SIAM J. Sci. Comput. 27, 1651–1668, 2006

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- LSC preconditioner
  - equivalent to

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} A & B^T \\ 0 & S \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ BA^{-1} & I \end{pmatrix}$$

  - $\Longrightarrow$: good right preconditioner

$$\begin{pmatrix} A & B^T \\ 0 & S \end{pmatrix}^{-1}$$

  - approximation of Schur complement based on (commutation) ansatz

$$A \left(D_{\text{lsc}}^{-1} B^T \right) \approx B^T A_{\text{pres}} \quad \Longleftrightarrow \quad D_{\text{lsc}}^{-1} B^T A_{\text{pres}}^{-1} \approx A^{-1} B^T$$

D_{lsc} – scaling matrix, diagonal matrix, positive diagonal entries

A_{pres} – discretization of convection-diffusion operator for pressure

$$\Longrightarrow \quad S = -BA^{-1}B^T \approx -BD_{\text{lsc}}^{-1} B^T A_{\text{pres}}^{-1} = S_{\text{lsc}}$$

- LSC preconditioner
  - determine A_{pres} by minimizing commutation error in least squares sense
  - approximation

$$S_{\mathrm{lsc}} = -BD_{\mathrm{lsc}}^{-1}B^T \left(BD_{\mathrm{lsc}}^{-1}AD_{\mathrm{lsc}}^{-1}B^T\right)^{-1} BD_{\mathrm{lsc}}^{-1}B^T$$

  - problem for preconditioner

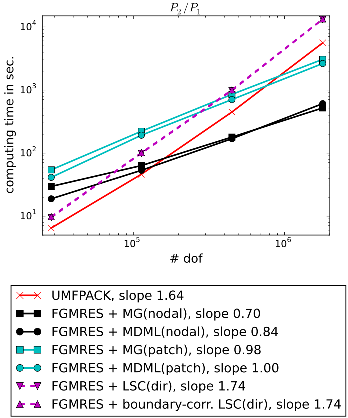
$$\begin{pmatrix} A & B^T \\ 0 & S_{\mathrm{lsc}} \end{pmatrix} \begin{pmatrix} \underline{v} \\ \underline{q} \end{pmatrix} = \begin{pmatrix} \underline{b}_v \\ \underline{b}_q \end{pmatrix}$$

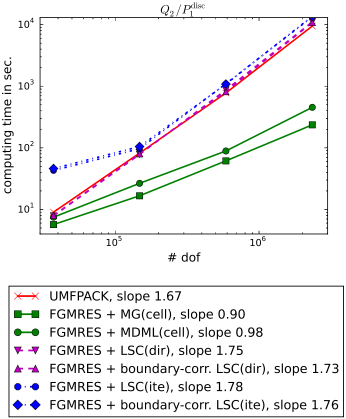
  - requires A^{-1} and inverse of Schur complement approximation

$$S_{\mathrm{lsc}}^{-1} = -\left(BD_{\mathrm{lsc}}^{-1}B^T\right)^{-1} \left(BD_{\mathrm{lsc}}^{-1}AD_{\mathrm{lsc}}^{-1}B^T\right) \left(BD_{\mathrm{lsc}}^{-1}B^T\right)^{-1}$$

    – $\left(BD_{\mathrm{lsc}}^{-1}B^T\right)^{-1}$ (scaled) Poisson problem
    – our implementation: compute $BD_{\mathrm{lsc}}^{-1}B^T$ explicitly $\Longrightarrow$ sparse direct solver can be applied

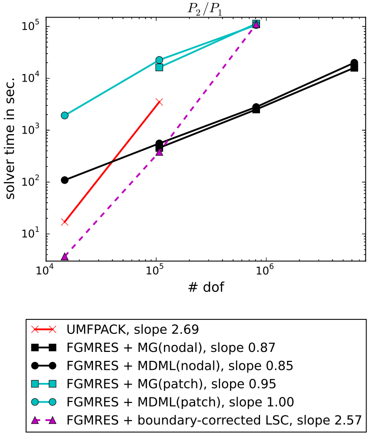
- numerical studies [1]
  - 2d stationary Navier–Stokes equations, flow around a cylinder

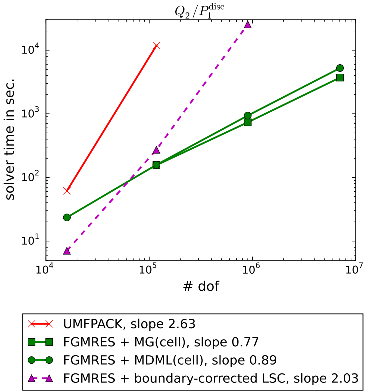




[1] Ahmed, Bartsch, J., Wilbrandt; CMAME 331, 492–513, 2018

- numerical studies [1]
  - 3d stationary Navier–Stokes equations, flow around a cylinder





[1] Ahmed, Bartsch, J., Wilbrandt; CMAME 331, 492–513, 2018

- numerical experience [1]
  - stationary Navier–Stokes equations
    - matrix A dominated by convective term
    - behavior depends somewhat on chosen pair of finite element spaces
    - sparse direct solver and LSC efficient only on coarse grids, in particular in 3d
    - on finer grids some multigrid preconditioner best
    - LSC preconditioner with approximate solution of system with A does not work
- numerical experience for time-dependent Navier–Stokes equations in Chapter 9

[1] Ahmed, Bartsch, J., Wilbrandt; CMAME 331, 492–513, 2018

8. The Time-Dependent Navier–Stokes Equations – Analysis

- continuous equation

$$\begin{array}{rcll} \partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p & = & \boldsymbol{f} & \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} & = & 0 & \text{in } (0,T]\times\Omega, \\ \boldsymbol{u}(0,\cdot) & = & \boldsymbol{u}_0 & \text{in } \Omega, \end{array}$$

with

$$\boldsymbol{u} = \boldsymbol{0} \text{ in } (0,T]\times\Gamma$$

- weak or variational formulation obtained by
  - multiply Navier–Stokes equations with a suitable test function $\boldsymbol{\varphi}$
  - integrate on $(0,T)\times\Omega$
  - apply integration by parts
- weak or variational formulation
  - let $\boldsymbol{f}\in L^2\left(0,T;V'\right)$ and $\boldsymbol{u}_0\in H_{\mathrm{div}}(\Omega)$
  - $\boldsymbol{u}$ is called weak or variational solution of the Navier–Stokes equations if
    - $\boldsymbol{u}$ satisfies

$$\int_0^T \Big[-\left(\boldsymbol{u},\partial_t\boldsymbol{\phi}\right)+\nu\left(\nabla\boldsymbol{u},\nabla\boldsymbol{\phi}\right)+\left(\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u},\boldsymbol{\phi}\right)\Big](\tau)\,d\tau$$

$$= \quad \int_0^T\langle\boldsymbol{f},\boldsymbol{\phi}\rangle_{V',V}(\tau)\,d\tau+\left(\boldsymbol{u}_0,\boldsymbol{\phi}\left(0,\cdot\right)\right)\quad\forall\ \boldsymbol{\phi}\in C_{0,\mathrm{div}}^{\infty}\left([0,T)\times\Omega\right).$$

    - $\boldsymbol{u}$ has the following regularity

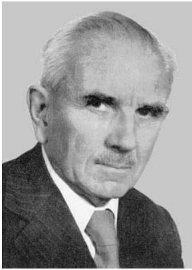
$$\boldsymbol{u}\in L^2\left(0,T;V_{\mathrm{div}}\right)\cap L^\infty\left(0,T;H_{\mathrm{div}}(\Omega)\right)$$

- properties
  - no time derivative with respect to $\boldsymbol{u}$
  - no second order space derivative with respect to $\boldsymbol{u}$
  - the pressure vanished because

$$\int_\Omega \nabla p \cdot \boldsymbol{\varphi}\, d\boldsymbol{x} = (\nabla p, \boldsymbol{\varphi}) = \int_\Gamma p(\boldsymbol{s})\, \underbrace{\boldsymbol{\varphi}(\boldsymbol{s})}_{=\boldsymbol{0}} \cdot \boldsymbol{n}(\boldsymbol{s})\, d\boldsymbol{s} - (p, \underbrace{\nabla \cdot \boldsymbol{\varphi}}_{=0}) = 0$$

- mathematical analysis
  - 2d: existence and uniqueness of weak solution, Leray (1933), Hopf (1951)
  - 3d: existence of weak solution, Leray (1933), Hopf (1951)

- Jean Leray (1906 – 1998) Eberhard Hopf (1902 – 1983)



Uniqueness of weak solution of 3d Navier–Stokes equations is open problem !

- principal idea of all existence proofs
  - consider simpler problem than Navier–Stokes equations
    - simpler problem has parameter such that in some limit Navier–Stokes equations are obtained
  - show existence and uniqueness of a solution of simpler problem
  - show that in a limit a subsequence of these solutions converges to a weak solution of the Navier–Stokes equations

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- Hopf (1951): simpler problems are Navier–Stokes equations in finite-dimensional subspace (Galerkin method)

- starting point: take $L^2(\Omega)$ orthonormal basis $\{\boldsymbol{v}_l\}_{l=1}^{\infty}$ of $C_{0,\text{div}}^{\infty}(\Omega)$
- finite-dimensional space

$$V_{\text{div}}^n = \text{span}\{\boldsymbol{v}_l^n\}_{l=1}^n \subset C_{0,\text{div}}^{\infty}(\Omega)$$

- equation in this space: Find $\boldsymbol{u}^n \in V_{\text{div}}^n$ such that

$$(\partial_t \boldsymbol{u}^n, \boldsymbol{v}^n) + (\nu \nabla \boldsymbol{u}^n, \nabla \boldsymbol{v}^n) + n(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}^n) = \langle \boldsymbol{f}, \boldsymbol{v}^n \rangle_{V',V} \quad \forall\, \boldsymbol{v}^n \in V_{\text{div}}^n$$

- ansatz for solution

$$\boldsymbol{u}^n\left(t,\boldsymbol{x}\right) = \sum_{l=1}^{n} \alpha_l^n(t)\boldsymbol{v}_l^n\left(\boldsymbol{x}\right)$$

- system of ordinary differential equations

$$\begin{aligned}
\frac{d\alpha_l^n}{dt} + \sum_{j=1}^{n} a_{lj}\alpha_j^n + \sum_{j,k=1}^{n} n_{ljk}\alpha_j^n\alpha_k^n &= f_l, \quad l=1,\ldots,n,\\
\alpha_l^n(0) &= u_{0l},, \quad l=1,\ldots,n
\end{aligned}$$

with

$$a_{lj} = \left(\nu\nabla\boldsymbol{v}_j^n, \nabla\boldsymbol{v}_l^n\right), \quad n_{ljk} = \left(\left(\boldsymbol{v}_j^n\cdot\nabla\right)\boldsymbol{v}_k^n, \boldsymbol{v}_l^n\right) = n\left(\boldsymbol{v}_j^n, \boldsymbol{v}_k^n, \boldsymbol{v}_l^n\right),$$
$$f_l = \langle \boldsymbol{f}, \boldsymbol{v}_l^n\rangle_{V',V}, \quad u_{0l} = \left(\boldsymbol{u}_0, \boldsymbol{v}_l^n\right)$$

- existence of unique solution: Theorem of Carathéodory (generalization of Theorem of Peano)

- prove weak convergence of subsequence of solution in the spaces present in definition of weak solution
- prove that nonlinear term of finite-dimensional problems converges to nonlinear term of Navier–Stokes equations (lengthy and technical)
- prove that limit of subsequence satisfies initial condition of Navier–Stokes equations

existence of a weak solution

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- prove that nonlinear term of finite-dimensional problems converges to nonlinear term of Navier–Stokes equations (lengthy and technical)
- prove that limit of subsequence satisfies initial condition of Navier–Stokes equations

existence of a weak solution

- some consequences
  - regularity for temporal derivative

$$\partial_t \boldsymbol{u} \in \begin{cases} L^2\left(0,T;V'\right) & \text{if } d=2, \\ L^{4/3}\left(0,T;V'\right) & \text{if } d=3 \end{cases}$$

- some consequences
  - energy inequality

$$\|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2 + 2\nu \int_0^t \|\nabla \boldsymbol{u}(\tau)\|_{L^2(\Omega)}^2 \, d\tau \leq \|\boldsymbol{u}(0)\|_{L^2(\Omega)}^2 + 2\int_0^t \langle \boldsymbol{f}, \boldsymbol{u}\rangle_{V'V}(\tau)\, d\tau$$

    - 2d: even energy equality
  - stability for all times t

$$\|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2 + \nu \|\nabla \boldsymbol{u}\|_{L^2(0,t;L^2(\Omega))}^2 \leq \|\boldsymbol{u}(0)\|_{L^2(\Omega)}^2 + \frac{1}{\nu} \|\boldsymbol{f}\|_{L^2(0,t;H^{-1}(\Omega))}^2$$

- uniqueness of a weak solution
  - 2d: by Sobolev imbedding

$$\boldsymbol{u} \in L^4\left(0, T; L^4(\Omega)\right)$$

    - is suffient for uniqueness of weak solution
  - 3d: with stronger regularity assumption, e.g., $\boldsymbol{u} \in L^8\left(0, T; L^4(\Omega)\right)$
  - 3d: generalization [1]

$$\boldsymbol{u} \in L^s\left(0, T; L^q(\Omega)\right) \quad \text{with} \quad s > 2, q > 3, \ \frac{2}{s} + \frac{3}{q} = 1$$

  - 3d: question is open

[1] Serrin; University of Wisconsin Press, Madison 69–98, 1963

- weak formulation: Find $\boldsymbol{u} : (0,T] \to V$, $p : (0,T] \to Q$ such that

$$(\partial_t \boldsymbol{u}, \boldsymbol{v}) + (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) + (\nabla \cdot \boldsymbol{u}, q) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V',V}$$

for all $(\boldsymbol{v}, q) \in V \times Q$ and $\boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \in H_{\mathrm{div}}(\Omega)$

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  for all $(\boldsymbol{v}, q) \in V \times Q$ and $\boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \in H_{\mathrm{div}}(\Omega)$
- $V^h \subset V$, $Q^h \subset Q$
- time-continuous finite element problem: Find $\boldsymbol{u}^h : (0,T] \to V^h$, $p^h : (0,T] \to Q^h$ such that

  $$\begin{aligned}\left(\partial_t \boldsymbol{u}^h, \boldsymbol{v}^h\right) + (\nu \nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h) + n\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right) & \\ - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) + \left(\nabla \cdot \boldsymbol{u}^h, q^h\right) &= \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle_{V',V}\end{aligned}$$

  for all $(\boldsymbol{v}^h, q^h) \in V^h \times Q^h$ and $\boldsymbol{u}^h(0, \boldsymbol{x}) = \boldsymbol{u}_0^h(\boldsymbol{x}) \in V^h$, approximation of $\boldsymbol{u}_0(\boldsymbol{x})$
  - Galerkin discretization
  - skew-symmetric form of the convective term

- existence, uniqueness, stability of finite element solution
  - same tools as for Galerkin method in proof from Hopf
- assumption for error analysis
  - data

$$\boldsymbol{f} \in L^2\left(0,T;V'\right), \quad \boldsymbol{u}_0 \in H_{\mathrm{div}}(\Omega), \quad \boldsymbol{u}_0^h \in V_{\mathrm{div}}^h$$

  - solution of continuous problem

$$\partial_t \boldsymbol{u} \in L^2\left(0,T;V'\right), \quad \nabla \boldsymbol{u} \in L^4\left(0,T;L^2\left(\Omega\right)\right), \quad p \in L^2\left(0,T;L^2\left(\Omega\right)\right)$$

$\Longrightarrow$ uniqueness of weak solution

- steps of the proof
  1. derivation of an error equation and splitting of the error
     - same as usual: subtract finite element problem from continuous problem

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  3. application of Gronwall's lemma
  4. application of the triangle inequality
     - same as usual

- Gronwall's lemma (in differential form):

  Let $T \in \mathbb{R}^+ \cup \infty$, $f \in W^{1,1}(0,T)$, and $g, \lambda \in L^1(0,T)$. Then

$$f'(t) \leq g(t) + \lambda(t) f(t) \quad a.e.\ in\ [0,T]$$

  implies for almost all $t \in [0,T]$

$$f(t) \leq \exp\left(\int_0^t \lambda(\tau)\, d\tau\right) f(0) + \int_0^t \exp\left(\int_s^t \lambda(\tau)\, d\tau\right) g(s)\, ds.$$

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- result of step 2: for all $q^h \in Q^h$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \frac{3\nu}{8}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 &\leq \frac{2}{\nu}\left\|\partial_t\boldsymbol{\eta}\right\|_{V'}^2 + \frac{2}{\nu}\left\|p - q^h\right\|_{L^2(\Omega)}^2 \\ &+ \frac{C}{\nu}\Big(\|\boldsymbol{\eta}\|_{L^2(\Omega)}\|\nabla\boldsymbol{\eta}\|_{L^2(\Omega)}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^2 + \left\|\boldsymbol{u}^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)}\|\nabla\boldsymbol{\eta}\|_{L^2(\Omega)}^2 \\ &+ \frac{C}{\nu^3}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^4\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2\end{aligned}$$

- integrate inequality in time
- assumptions and stability estimates show, e.g.,

$$\int_0^t \|\boldsymbol{\eta}\|_{L^2(\Omega)} \|\nabla\boldsymbol{\eta}\|_{L^2(\Omega)} \|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^2 \, d\tau < \infty$$

and

$$\int_0^t \left\|\boldsymbol{u}^h\right\|_{L^2(\Omega)} \left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)} \|\nabla\boldsymbol{\eta}\|_{L^2(\Omega)}^2 \, d\tau < \infty$$

$\Longrightarrow$ assumptions of Gronwall's lemma satisfied

- final error estimate: $I_{\mathrm{St}}^h \boldsymbol{u}(t)$ – appropriate projection

$$
\begin{aligned}
&\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h\right)(t)\right\|_{L^2(\Omega)}^2+\nu\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(0,t;L^2(\Omega))}^2 \\
&\leq \quad C\Bigg\{\left\|\left(\boldsymbol{u}-I_{\mathrm{St}}^h \boldsymbol{u}\right)(t)\right\|_{L^2(\Omega)}^2+\nu\left\|\nabla\left(\boldsymbol{u}-I_{\mathrm{St}}^h \boldsymbol{u}\right)\right\|_{L^2(0,t;L^2(\Omega))}^2 \\
&\qquad +\exp\left(\frac{C}{\nu^3}\|\nabla \boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^4\right)\Bigg[\left\|\boldsymbol{u}_0^h-I_{\mathrm{St}}^h \boldsymbol{u}(0)\right\|_{L^2(\Omega)}^2 \\
&\qquad +\frac{1}{\nu}\left(\left\|\partial_t\left(\boldsymbol{u}-I_{\mathrm{St}}^h \boldsymbol{u}\right)\right\|_{L^2(0,t;V')}^2\right. \\
&\qquad +\left\|\nabla\left(\boldsymbol{u}-I_{\mathrm{St}}^h \boldsymbol{u}\right)\right\|_{L^4(0,t;L^2(\Omega))}^2\|\nabla \boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^2 \\
&\qquad \left.+\inf_{q^h\in L^2(0,t;Q^h)}\left\|p-q^h\right\|_{L^2(0,t;L^2(\Omega))}^2\right) \\
&\qquad +\frac{1}{\nu^{3/2}}\left(\left\|\boldsymbol{u}_0^h\right\|_{L^2(\Omega)}^2+\frac{1}{\nu}\|\boldsymbol{f}\|_{L^2(0,t;V')}^2\right)\left\|\nabla\left(\boldsymbol{u}-I_{\mathrm{St}}^h \boldsymbol{u}\right)\right\|_{L^4(0,t;L^2(\Omega))}^2\Bigg]\Bigg\}
\end{aligned}
$$

- final error estimate
  - factor very (!) large (unrealistic)

$$\exp\left(\frac{C}{\nu^3}\left\|\nabla \boldsymbol{u}\right\|^4_{L^4(0,t;L^2(\Omega))}\right)$$

  - useless error bound for practice

[1] J., Knobloch, Novo; Comput. Vis. Sci. 19, 47–63, 2018

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  - modifications with other (higher) regularity assumptions on solution possible
    - exponential depends only on ν^{-1}
  - for weakly divergence-free pairs of spaces
    - simplest form of convective term can be used in analysis

$$n_{\text{conv}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w})$$

    - exponential does not depend on inverse powers of ν

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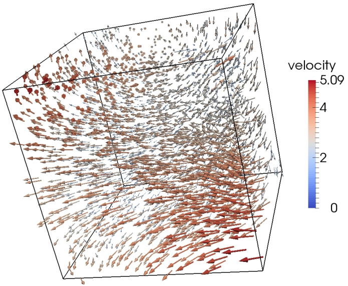
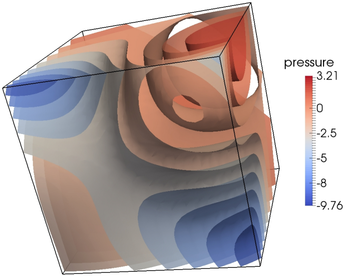
$$n_{\text{conv}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})$$

    - exponential does not depend on inverse powers of ν
- estimate for pressure: lengthy and very technical
- survey of open problems in [1]

[1] J., Knobloch, Novo; Comput. Vis. Sci. 19, 47–63, 2018

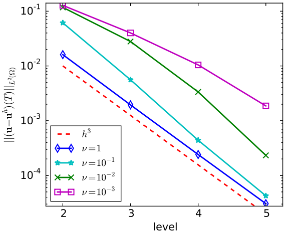
- example with analytical solution (Beltrami flow)

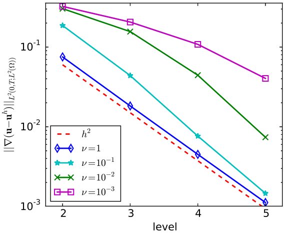


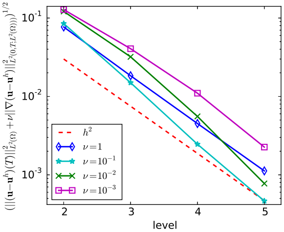


  - Q_2/Q_1
  - very small time step (temporal error negligible)

- example with analytical solution (Beltrami flow): convergence of velocity errors







9. The Time-Dependent Navier–Stokes Equations – Schemes

- implicit θ-schemes as semi discretization in time
  - $\Delta t_{n+1} = t_{n+1} - t_n$
  - subscript k for quantities at time level k

$$
\begin{aligned}
\boldsymbol{u}_{k+1} + \theta_1 \Delta t_{n+1} [-\nu \Delta \boldsymbol{u}_{k+1} + (\boldsymbol{u}_{k+1} \cdot \nabla)\, \boldsymbol{u}_{k+1}] + \Delta t_{k+1} \nabla p_{k+1} \\
= \quad & \boldsymbol{u}_k - \theta_2 \Delta t_{n+1} [-\nu \nabla \cdot \Delta \boldsymbol{u}_k + (\boldsymbol{u}_k \cdot \nabla)\, \boldsymbol{u}_k] + \theta_3 \Delta t_{n+1} \boldsymbol{f}_k \\
& + \theta_4 \Delta t_{n+1} \boldsymbol{f}_{k+1}, \\
\nabla \cdot \boldsymbol{u}_{k+1} \quad = \quad & 0,
\end{aligned}
$$

- implicit θ-schemes as semi discretization in time
  - $\Delta t_{n+1} = t_{n+1} - t_n$
  - subscript k for quantities at time level k

$$
\begin{aligned}
\boldsymbol{u}_{k+1} + \theta_1 \Delta t_{n+1}\left[-\nu\Delta \boldsymbol{u}_{k+1} + (\boldsymbol{u}_{k+1}\cdot\nabla)\,\boldsymbol{u}_{k+1}\right] + \Delta t_{k+1}\nabla p_{k+1} & \\
= \quad & \boldsymbol{u}_k - \theta_2 \Delta t_{n+1}\left[-\nu\nabla\cdot\Delta \boldsymbol{u}_k + (\boldsymbol{u}_k\cdot\nabla)\,\boldsymbol{u}_k\right] + \theta_3\Delta t_{n+1}\boldsymbol{f}_k \\
& + \theta_4 \Delta t_{n+1}\boldsymbol{f}_{k+1}, \\
\nabla\cdot\boldsymbol{u}_{k+1} \quad = \quad & 0,
\end{aligned}
$$

- one-step θ-schemes: $n = k$

	θ_1	θ_2	θ_3	θ_4	t_k	t_{k+1}	Δt_{k+1}	order
forward Euler scheme	0	1	1	0	t_n	t_{n+1}	Δt_{n+1}	
backward Euler scheme (BWE)	1	0	0	1	t_n	t_{n+1}	Δt_{n+1}	1
Crank–Nicolson scheme (CN)	0.5	0.5	0.5	0.5	t_n	t_{n+1}	Δt_{n+1}	2

- fractional-step θ-scheme [1]
  - three-step scheme
  - two variants

$$\theta = 1 - \frac{\sqrt{2}}{2}, \quad \tilde{\theta} = 1 - 2\theta, \quad \tau = \frac{\tilde{\theta}}{1-\theta}, \quad \eta = 1 - \tau$$

	θ_1	θ_2	θ_3	θ_4	t_k	t_{k+1}	Δt_{k+1}	order
FS0	$\tau\theta$	$\eta\theta$	$\eta\theta$	$\tau\theta$	t_n	$t_n + \theta\Delta t_{n+1}$	$\theta\Delta t_{n+1}$	
	$\eta\tilde{\theta}$	$\tau\tilde{\theta}$	$\tau\tilde{\theta}$	$\eta\tilde{\theta}$	$t_n + \theta\Delta t_{n+1}$	$t_{n+1} - \theta\Delta t_{n+1}$	$\tilde{\theta}\Delta t_{n+1}$	2
	$\tau\theta$	$\eta\theta$	$\eta\theta$	$\tau\theta$	$t_{n+1} - \theta\Delta t_{n+1}$	t_{n+1}	$\theta\Delta t_{n+1}$	
FS1	$\tau\theta$	$\eta\theta$	θ	0	t_n	$t_n + \theta\Delta t_{n+1}$	$\theta\Delta t_{n+1}$	
	$\eta\tilde{\theta}$	$\tau\tilde{\theta}$	0	$\tilde{\theta}$	$t_n + \theta\Delta t_{n+1}$	$t_{n+1} - \theta\Delta t_{n+1}$	$\tilde{\theta}\Delta t_{n+1}$	2
	$\tau\theta$	$\eta\theta$	θ	0	$t_{n+1} - \theta\Delta t_{n+1}$	t_{n+1}	$\theta\Delta t_{n+1}$	

[1] Bristeau, Glowinski, Periaux: Finite elements in physics, North-Holland, 73–187, 1986

- popular approaches: BWE, CN, BDF2
- stability
  - BWE, FS0, FS1, BDF2: strongly A-stable
  - CN: A-stable
- FS1 less expensive than FS0 if computation of right-hand side costly

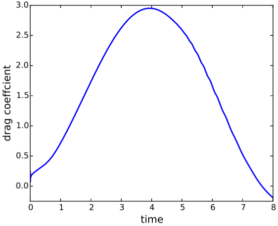
- popular approaches: BWE, CN, BDF2
- stability
  - BWE, FS0, FS1, BDF2: strongly A-stable
  - CN: A-stable
- FS1 less expensive than FS0 if computation of right-hand side costly
- number of papers with finite element error estimates available
  - proofs become long
  - same techniques as for continuous-in-time problem + discrete Gronwall's lemma

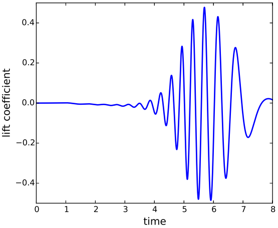
- implementation
  - goes along the same lines as for Stokes and Navier–Stokes equations
  - additional loop over time instances needed
  - temporal derivative leads to mass matrix M: symmetric, positive definite
  - principal form of the system

$$\begin{pmatrix} M + \theta\Delta t_{n+1}A & \theta\Delta t_{n+1}B^T \\ \Delta t_{n+1}B & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_{n+1} \\ \underline{p}_{n+1} \end{pmatrix} = \begin{pmatrix} \underline{\text{rhs}} \\ \underline{0} \end{pmatrix}$$

    – mass matrix dominant for small time steps (good property!)
    – our experience: scaling of discrete continuity equation very helpful for efficiency of solvers

- flow around a cylinder
  - reference curves for drag and lift [1]





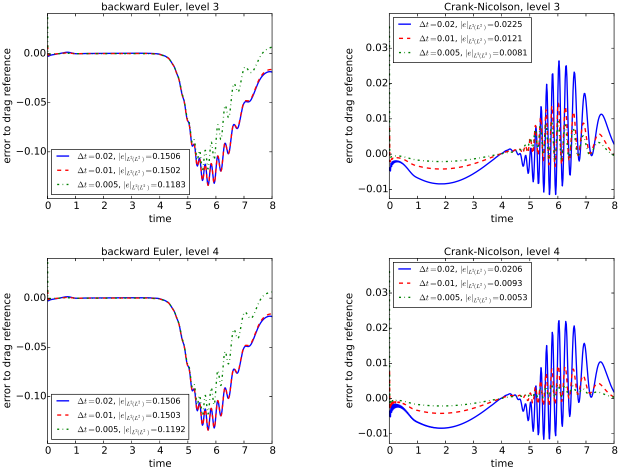
[1] J., Rang, CMAME 199, 514–524, 2010

- refinement in space with Q_2/P_1^{disc}

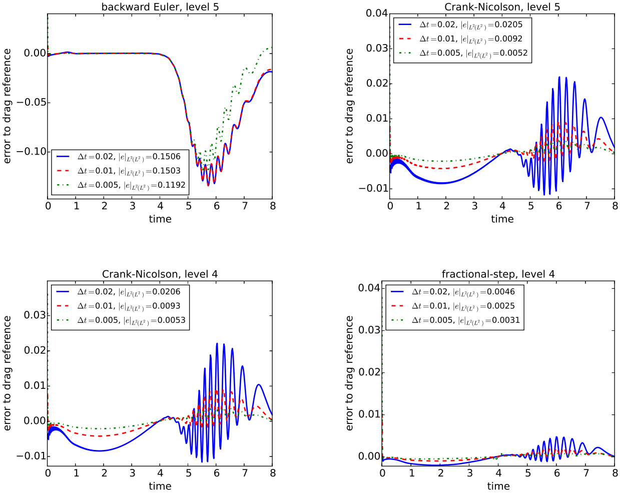
	P_2/P_1			Q_2/P_1^{disc}		
level	velocity	pressure	all	velocity	pressure	all
3	25 408	3248	28 656	27 232	9984	37 216
4	100 480	12 704	113 184	107 712	39 936	147 648
5	399 616	50 240	449 856	428 416	159 744	588 160

- refinement in time: $\Delta t \in \{0.02, 0.01, 0.005\}$

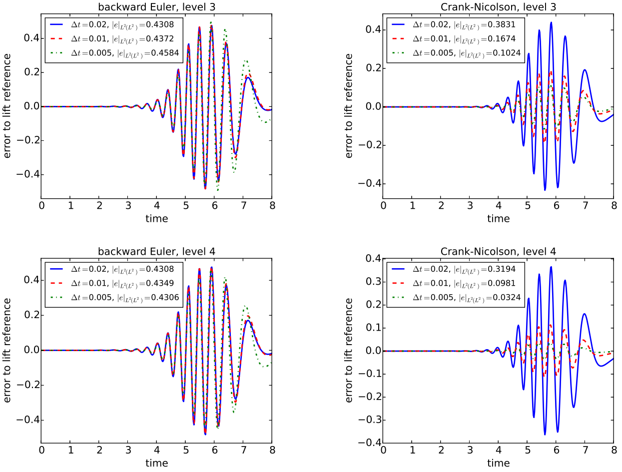
- error to the reference curve for the drag coefficient



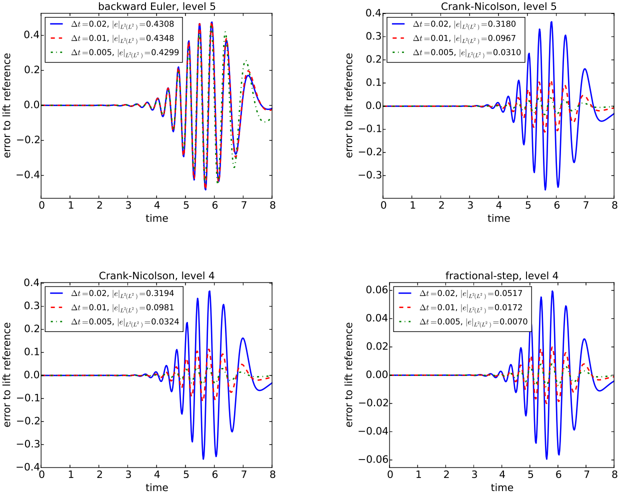
- error to the reference curve for the drag coefficient



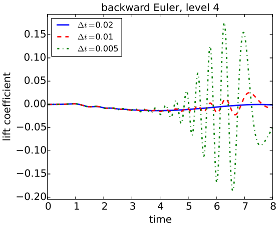
- error to the reference curve for the lift coefficient

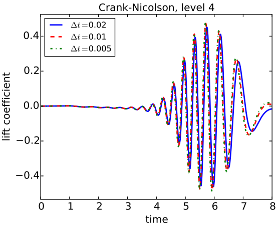


- error to the reference curve for the lift coefficient



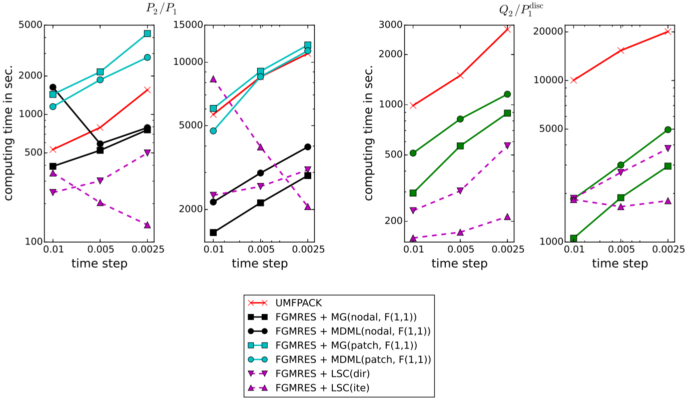
- temporal evolution of lift coefficient





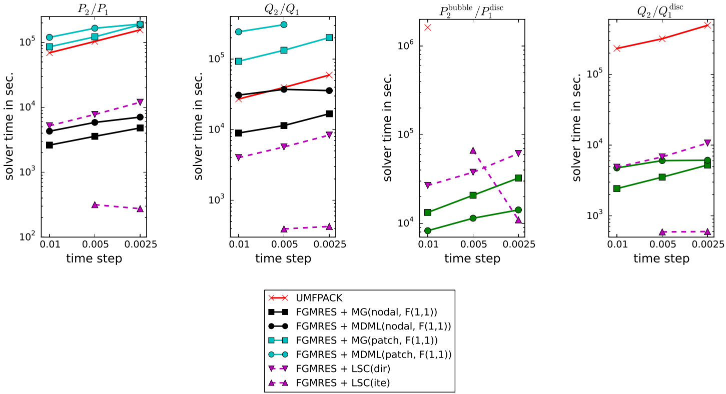
- BWE much to inaccurate (dissipative)

- numerical studies [1]
- same solvers as for stationary Navier–Stokes equations
  - 2d time-dependent Navier–Stokes equations, different refinement levels



[1] Ahmed, Bartsch, J., Wilbrandt; CMAME 331, 492–513, 2018

- numerical studies [1]
- same solvers as for stationary Navier–Stokes equations
  - 3d time-dependent Navier–Stokes equations



[1] Ahmed, Bartsch, J., Wilbrandt; CMAME 331, 492–513, 2018

- numerical studies [1]
  - time-dependent Navier–Stokes equations
    - mass matrix dominates discrete momentum equation
    - LSC with iterative solution of system with A (norm of residual reduced by a factor of 100 with BiCGStab) best for small time steps
    - expensive setup & factorization of Poisson-type matrix only in first time step
    - number of BiCGStab iterations decreases with smaller time steps
    - for largest time step often some multigrid method best

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    - expensive setup & factorization of Poisson-type matrix only in first time step
    - number of BiCGStab iterations decreases with smaller time steps
    - for largest time step often some multigrid method best
- recent experience: LSC works well in parallel framework with small number of processors
  - 50 processors (largest machine of the institute)
  - MPI
  - MUMPS for solving Poisson-type problem

[1] Ahmed, Bartsch, J., Wilbrandt; CMAME 331, 492–513, 2018

- projection method
  - motivation: schemes without need to solve (nonlinear) saddle point problems
  - survey in [1]

[1] Guermond, Minev, Shen, CMAME 195, 6011–6045, 2006

- idea: decoupled NSE to obtain separate equations for velocity and pressure
  - approximation of time derivative given (q-step scheme)

$$\partial_t \boldsymbol{u}\left(t_{n+1}\right) \approx \frac{1}{\Delta t}\left(\tau_q \boldsymbol{u}_{n+1} + \sum_{i=0}^{q-1} \tau_j \boldsymbol{u}_{n-j}\right), \quad \sum_{i=0}^{q} \tau_j = 0$$

  - equation for intermediate velocity: given $\hat{p}$ or $\nabla \hat{p}$

$$\frac{1}{\Delta t}\left(\tau_q \tilde{\boldsymbol{u}}_{n+1} + \sum_{i=0}^{q-1} \tau_j \boldsymbol{u}_{n-j}\right) - \nu \Delta \tilde{\boldsymbol{u}} + (\tilde{\boldsymbol{u}} \cdot \nabla)\tilde{\boldsymbol{u}} = \boldsymbol{f} - \nabla \hat{p} \quad \text{in } \Omega$$

  - correction step for divergence-free velocity

$$\begin{aligned} \frac{1}{\Delta t}\left(\tau_q \boldsymbol{u}_{n+1} - \tau_q \tilde{\boldsymbol{u}}_{n+1}\right) + \nabla \varphi\left(\tilde{\boldsymbol{u}}\right) + \nabla p &= \nabla \hat{p} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}_{n+1} &= 0 \quad \text{in } \Omega \end{aligned}$$

$\varphi(\cdot)$ – given function

- velocity computed in projection step is $L^2(\Omega)$ projection of $\tilde{\boldsymbol{u}}_{n+1}$ into

$$H_{\mathrm{div}}(\Omega) = \left\{\boldsymbol{v} \in L^2(\Omega),\ \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\ :\ \nabla\cdot\boldsymbol{v} = 0 \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\}$$

- non-incremental pressure-correction scheme
  - $\hat{p} = 0, \varphi(\cdot) = 0$
  - proposed in [1,2]
  - with backward Euler
- intermediate velocity

$$\tilde{\boldsymbol{u}}_{n+1} + \Delta t_{n+1} \left(-\nu \Delta \tilde{\boldsymbol{u}}_{n+1} + (\tilde{\boldsymbol{u}}_n \cdot \nabla) \tilde{\boldsymbol{u}}_{n+1} \right) = \boldsymbol{u}_n + \Delta t_{n+1} \boldsymbol{f}_{n+1} \quad \text{in } \Omega$$

with $\tilde{\boldsymbol{u}}_{n+1} = \boldsymbol{0}$ on Γ

- projection step

$$\begin{aligned} \boldsymbol{u}_{n+1} + \Delta t_{n+1} \nabla p_{n+1} &= \tilde{\boldsymbol{u}}_{n+1} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}_{n+1} &= 0 && \text{in } \Omega, \\ \boldsymbol{u}_{n+1} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma \end{aligned}$$

[1] Chorin, Math. Comp. 22, 745–762, 1968

[2] Temam, Arch. Rational Mech. Anal. 33, 377–385, 1969

- non-incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\nabla \cdot \nabla p_{n+1} = \Delta p_{n+1} = \frac{1}{\Delta t_{n+1}} \nabla \cdot \tilde{\boldsymbol{u}}_{n+1}$$

  - Poisson equation for the pressure
  - boundary condition

$$\nabla p_{n+1} \cdot \boldsymbol{n} = -\frac{1}{\Delta t_{n+1}} \left(\boldsymbol{u}_{n+1} - \tilde{\boldsymbol{u}}_{n+1}\right) \cdot \boldsymbol{n} = 0$$

- non-incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\nabla \cdot \nabla p_{n+1} = \Delta p_{n+1} = \frac{1}{\Delta t_{n+1}} \nabla \cdot \tilde{\boldsymbol{u}}_{n+1}$$

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$$\nabla p_{n+1} \cdot \boldsymbol{n} = -\frac{1}{\Delta t_{n+1}} \left(\boldsymbol{u}_{n+1} - \tilde{\boldsymbol{u}}_{n+1} \right) \cdot \boldsymbol{n} = 0$$

- error estimates: $(\overline{\boldsymbol{u}}, \overline{p})$ result of projection step

$$\|p - \overline{p}\|_{l^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{l^\infty(0,T;H^1(\Omega))} \le C\left(\boldsymbol{u}, p, T\right) \Delta t^{1/2}$$

if in addition domain has regularity property

$$\|\boldsymbol{u} - \overline{\boldsymbol{u}}\|_{l^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{l^\infty(0,T;L^2(\Omega))} \le C\left(\boldsymbol{u}, p, T\right) \Delta t$$

- non-incremental pressure-correction scheme (cont.)
  - inf-sup stable finite elements not necessary
  - however, spurious oscillations may appear if the time step becomes too small
  - low orders of convergence
  - splitting error is $\mathcal{O}(\Delta t) \Longrightarrow$ first order time stepping scheme sufficient
  - artificial Neumann boundary condition for the pressure induces a numerical boundary layer

- standard incremental pressure-correction scheme
  - $\hat{p} = p_n$, $\varphi(\cdot) = 0$
  - with BDF2
- intermediate velocity

$$\begin{aligned} 3\tilde{\boldsymbol{u}}_{n+1} + 2\Delta t \left(-\nu\Delta\tilde{\boldsymbol{u}}_{n+1} + (\tilde{\boldsymbol{u}}_n\cdot\nabla)\tilde{\boldsymbol{u}}_{n+1}\right) & \\ = \quad 4\boldsymbol{u}_n - \boldsymbol{u}_{n-1} + 2\Delta t\left(\boldsymbol{f}_{n+1} - \nabla p_n\right) & \quad \text{in } \Omega, \end{aligned}$$

with $\tilde{\boldsymbol{u}}_{n+1} = \mathbf{0}$ on Γ

- projection step

$$\begin{aligned} 3\boldsymbol{u}_{n+1} + 2\Delta t\nabla\left(p_{n+1} - p_n\right) &= 3\tilde{\boldsymbol{u}}_{n+1} && \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}_{n+1} &= 0 && \text{in } \Omega, \\ \boldsymbol{u}_{n+1}\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma \end{aligned}$$

- standard incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta\left(p_{n+1} - p_n\right) = \frac{3}{2\Delta t}\nabla \cdot \tilde{\boldsymbol{u}}_{n+1} \quad \text{in } \Omega$$

  - Poisson equation for the pressure update
  - boundary condition

$$\nabla\left(p_{n+1} - p_n\right) \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma$$

- standard incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta\left(p_{n+1}-p_n\right)=\frac{3}{2\Delta t}\nabla\cdot\tilde{\boldsymbol{u}}_{n+1}\quad \text{in } \Omega$$

  - Poisson equation for the pressure update
  - boundary condition

$$\nabla\left(p_{n+1}-p_n\right)\cdot\boldsymbol{n}=0\quad \text{on } \Gamma$$

- error estimates, with appropriate initial step, $(\overline{\boldsymbol{u}},\overline{p})$ result of projection step

$$\|p-\overline{p}\|_{l^\infty(0,T;L^2(\Omega))}+\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{l^\infty(0,T;H^1(\Omega))}\le C\left(\boldsymbol{u},p,T\right)\Delta t$$

  if in addition domain has regularity property

$$\|\boldsymbol{u}-\overline{\boldsymbol{u}}\|_{l^\infty(0,T;L^2(\Omega))}+\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{l^2(0,T;L^2(\Omega))}\le C\left(\boldsymbol{u},p,T\right)\Delta t^2$$

- standard incremental pressure-correction scheme (cont.)
  - similar estimates for Crank–Nicolson scheme
  - splitting error is $\mathcal{O}\left(\Delta t^2\right) \Longrightarrow$ second order time stepping scheme sufficient
  - artificial Neumann boundary condition for the pressure induces a numerical boundary layer
  - non inf-sup stable pairs of finite element spaces need stabilization
    - consider steady-state solution, then problem for $\tilde{\boldsymbol{u}}_{n+1}$ is of saddle point type

- rotational incremental pressure-correction scheme
  - $\hat{p} = p_n, \varphi(\tilde{\boldsymbol{u}}) = \nu \nabla \cdot \tilde{\boldsymbol{u}}_{n+1}$
  - with BDF2
- intermediate velocity

$$
\begin{aligned}
3\tilde{\boldsymbol{u}}_{n+1} + 2\Delta t\,(-\nu\Delta\tilde{\boldsymbol{u}}_{n+1} + (\tilde{\boldsymbol{u}}_n \cdot \nabla)\tilde{\boldsymbol{u}}_{n+1}) \\
= \quad 4\boldsymbol{u}_n - \boldsymbol{u}_{n-1} + 2\Delta t\left(\boldsymbol{f}_{n+1} - \nabla p_n\right) \quad \text{in } \Omega,
\end{aligned}
$$

with $\tilde{\boldsymbol{u}}_{n+1} = \boldsymbol{0}$ on Γ

- projection step

$$
\begin{aligned}
3\boldsymbol{u}_{n+1} + 2\Delta t\nabla\left(p_{n+1} - p_n\right) &= 3\tilde{\boldsymbol{u}}_{n+1} - 2\nu\Delta t\nabla\left(\nabla \cdot \tilde{\boldsymbol{u}}_{n+1}\right) && \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u}_{n+1} &= 0 && \text{in } \Omega, \\
\boldsymbol{u}_{n+1} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma
\end{aligned}
$$

- rotational incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta \tilde{p}_n = \frac{3}{2\Delta t} \nabla \cdot \tilde{\boldsymbol{u}}_{n+1} \quad \text{with} \quad \tilde{p}_n = p_{n+1} - p_n + \nu \nabla \cdot \tilde{\boldsymbol{u}}_{n+1}$$

  - Poisson equation for the modified pressure
  - boundary condition

$$\nabla p_{n+1} \cdot \boldsymbol{n} = \left(\boldsymbol{f}_{n+1} - \nu \nabla \times \nabla \times \boldsymbol{u}_{n+1} \right) \cdot \boldsymbol{n} \text{ on } \Gamma$$

- rotational incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta \tilde{p}_n = \frac{3}{2\Delta t} \nabla \cdot \tilde{\boldsymbol{u}}_{n+1} \quad \text{with} \quad \tilde{p}_n = p_{n+1} - p_n + \nu \nabla \cdot \tilde{\boldsymbol{u}}_{n+1}$$

  - Poisson equation for the modified pressure
  - boundary condition

$$\nabla p_{n+1} \cdot \boldsymbol{n} = \left(\boldsymbol{f}_{n+1} - \nu \nabla \times \nabla \times \boldsymbol{u}_{n+1} \right) \cdot \boldsymbol{n} \text{ on } \Gamma$$

- error estimates, with appropriate initial step, $(\overline{\boldsymbol{u}}, \overline{p})$ result of projection step

$$\begin{aligned} &\| p - \overline{p} \|_{l^2(0,T;L^2(\Omega))} + \| \boldsymbol{u} - \tilde{\boldsymbol{u}} \|_{l^2(0,T;H^1(\Omega))} \\ &+ \| \boldsymbol{u} - \overline{\boldsymbol{u}} \|_{l^2(0,T;H^1(\Omega))} \le C\left(\boldsymbol{u}, p, T\right) \Delta t^{3/2} \end{aligned}$$

  if in addition domain has regularity property

$$\| \boldsymbol{u} - \overline{\boldsymbol{u}} \|_{l^2(0,T;L^2(\Omega))} + \| \boldsymbol{u} - \tilde{\boldsymbol{u}} \|_{l^2(0,T;L^2(\Omega))} \le C\left(\boldsymbol{u}, p, T\right) \Delta t^{2}$$

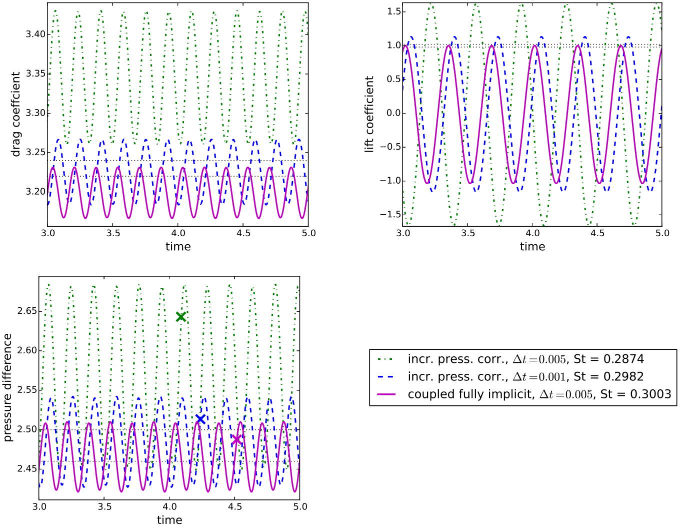
- rotational incremental pressure-correction scheme (cont.)
  - equivalent formulation of velocity step

$$
\begin{aligned}
3\boldsymbol{u}_{n+1} + 2\Delta t \left(\nu \nabla \times \nabla \times \boldsymbol{u}_{n+1} + (\tilde{\boldsymbol{u}}_n \cdot \nabla)\tilde{\boldsymbol{u}}_{n+1} + \nabla p_{n+1}\right) & \\
= \; 4\boldsymbol{u}_n - \boldsymbol{u}_{n-1} + 2\Delta t \boldsymbol{f}_{n+1} & \quad \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u}_{n+1} = \; 0 & \quad \text{in } \Omega
\end{aligned}
$$

  - boundary condition for the pressure is consistent, can be derived from the Navier–Stokes equations

- only $\tilde{\boldsymbol{u}}_{n+1}$ needed in implementation
- our experience with non-incremental and standard incremental scheme: very inaccurate at boundaries (bad drag and lift coefficients)

- periodic vortex street in 2D



10. Outlook: Simulation of Turbulent Flows

- continuous equation: incompressible Navier–Stokes equations

$$\begin{aligned} \partial_t \boldsymbol{u} - 2\mathrm{Re}^{-1}\nabla\cdot\mathbb{D}(\boldsymbol{u}) + \nabla\cdot(\boldsymbol{u}\boldsymbol{u}^T) + \nabla p &= \mathbf{f} && \text{in } (0,T]\times\Omega \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } [0,T]\times\Omega \\ \boldsymbol{u}(0,\boldsymbol{x}) &= \boldsymbol{u}_0 && \text{in } \Omega \end{aligned}$$

+ boundary conditions

- turbulent flows: Re very large

- continuous equation: incompressible Navier–Stokes equations

$$
\begin{aligned}
\partial_t \boldsymbol{u} - 2\mathrm{Re}^{-1}\nabla\cdot\mathbb{D}(\boldsymbol{u}) + \nabla\cdot(\boldsymbol{u}\boldsymbol{u}^T) + \nabla p &= \mathbf{f} && \text{in } (0,T]\times\Omega \\
\nabla\cdot\boldsymbol{u} &= 0 && \text{in } [0,T]\times\Omega \\
\boldsymbol{u}(0,\boldsymbol{x}) &= \boldsymbol{u}_0 && \text{in } \Omega
\end{aligned}
$$

+ boundary conditions

- turbulent flows: Re very large
- There is no exact definition of what is a turbulent flow !

- model: incompressible Navier–Stokes equations

$$\begin{aligned}
\partial_t \boldsymbol{u} - 2\mathrm{Re}^{-1}\nabla\cdot\mathbb{D}(\boldsymbol{u}) + \nabla\cdot(\boldsymbol{u}\boldsymbol{u}^T) + \nabla p &= \boldsymbol{f} && \text{in } (0,T]\times\Omega\\
\nabla\cdot\boldsymbol{u} &= 0 && \text{in } (0,T]\times\Omega\\
\boldsymbol{u}(0,\boldsymbol{x}) &= \boldsymbol{u}_0 && \text{in } \Omega\\
+ \text{ boundary conditions} &
\end{aligned}$$

- turbulent flows: Re very large
  - river flows
  - storms
  - flow around obstacles, e.g., cars
  - ⋮
- There is no exact definition of what is a turbulent flow !

- possess flow structures of very different size
  - hurricane Katrina (2005)

  - some large eddies (scales), many very small eddies (scales)

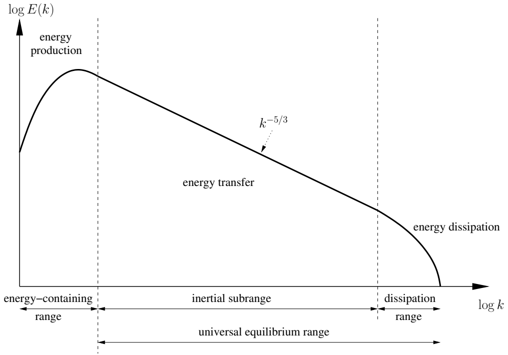
- Richardson energy cascade [1]: energy is transported in the mean from large to smaller eddies

  - start of cascade: kinetic energy introduced into flow by productive mechanisms at largest scale
  - inner cascade: transmitting energy to smaller and smaller scales by processes not depending on molecular viscosity
  - end of cascade: molecular viscosity enforcing dissipation of kinetic energy at smallest scales

- smallest scales important for physics of the flow

[1] L.F. Richardson; *Weather Prediction by Numerical Process*, Cambridge University Press, 1922

- energy of scales in wave number space (Fourier space)



- k – wave number
- $E(k)$ – turbulent kinetic energy of modes with wave number k
- $k^{-5/3}$ – law of energy spectrum: $E(k) \sim \epsilon^{2/3} k^{-5/3}$

- Kolmogorov [1]:

  energy is dissipated from eddies of size

$$\lambda \sim \mathrm{Re}^{-3/4}$$

  Kolmogorov scale

[1] A. Kolmogorov; C. R. (Doklady) Acad. Sci. URSS (N.S.) 30, 301–305, 1941

- Direct Numerical Simulation (DNS) aims to simulate all persisting eddies
  - $\Omega = (0,1)^3 \implies L = 1$
  - approx 10^7 cubic mesh cells ($\approx 215^3$)
  - low order method (mesh width $\approx$ resolution of discretization)
  - $\implies \lambda \approx 1/215$
  - $\implies$ Re ≈ 1290
- applications: Reynolds numbers larger by orders of magnitude

Direct Numerical Simulation not feasible !

- only resolved scales can be simulated
- transition from resolved to unresolved scales usually in inertial subrange

- smallest scales in 2d flows [1]: $\lambda \sim \text{Re}^{-1/2}$
- vortex stretching
  - vorticity: $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$
  - neglect viscous term for large Reynolds numbers

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} \approx \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$$

  - equation of infinitesimal line element of material
  - if $\nabla \boldsymbol{u}$ acts to stretch the line element than $|\boldsymbol{\omega}|$ will be stretched, too $\Longrightarrow$ vortex stretching, important feature of turbulent flows
  - 2d: right-hand side vanishes $\Longrightarrow$ no vortex stretching
- 2d high Reynolds number flows create large structures (eddies)
- 3d turbulent flows destroy large structures (eddies)

2d flows at high Reynolds number are qualitatively different from 3d turbulent flows

[1] R.H. Kraichnan; Physics of Fluids 10, 1417-1423, 1967

- direct simulation not possible
- only large scales can be simulated
- physics: (very) small scales important, have to be taken into account
- Main questions for turbulence modeling (simulation of turbulent flows):
  - How to define large?
    - spatial averaging $\Longrightarrow$ Large Eddy Simulation (LES)
    - projection in appropriate function spaces $\Longrightarrow$ Variational Multiscale (VMS) Methods
  - How to model the impact of the small scales onto the large scales?
    - several proposals and dozens of variants
    - here: Smagorinsky model (most popular)
- turbulence models should be less complex than the Navier–Stokes equations

- main idea in LES: large scales defined by averaging in space (convolution with filter function)
  - filter out small flow structures
  - damp high wave numbers

- main idea in LES: large scales defined by averaging in space (convolution with filter function)
  - filter out small flow structures
  - damp high wave numbers
- two-scale decomposition of the flow: large and unresolved scales

$$\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u}', \quad p = \overline{p} + p'$$

  - $\overline{\boldsymbol{u}}$, $\overline{p}$: large scales
  - $\boldsymbol{u}', p'$: subgrid scales
- goal of LES : approximate $\overline{\boldsymbol{u}}$, $\overline{p}$ $\Longrightarrow$ one needs equations for $\overline{\boldsymbol{u}}$, $\overline{p}$

- derivation of space-averaged Navier–Stokes equations (literature) :
  - filter Navier–Stokes equations with filter function g

$$g * (\nabla \cdot \boldsymbol{u}) = \overline{\nabla \cdot \boldsymbol{u}}$$

  - assume that convolution and differentiation commute

$$g * (\nabla \cdot \cdot) = \nabla \cdot (g * \cdot)$$

  - commute both operators

$$g * (\nabla \cdot \boldsymbol{u}) = \nabla \cdot (g * \boldsymbol{u}) = \nabla \cdot \overline{\boldsymbol{u}}$$

$\Longrightarrow$ expression for $\overline{\boldsymbol{u}}$

- assumption on commutation only valid for constant filter width and away from boundaries
- mathematical analysis [1,2,3] shows that commutation error is not negligible
- practice: no approach to incorporate commutation errors, they are simply neglected

[1] Dunca, J., Layton: Contributions to Current Challenges in Mathematical Fluid Mechanics, Birkhäuser Verlag, 53 – 78, 2004

[2] Berselli, J.: Math. Methods Appl. Sci. 29, 1709 – 1719, 2006

[3] Berselli, Grisanti, J.: J. Comp. Appl. Math. 206, 1027 – 1045, 2007

- assumption on commutation only valid for constant filter width and away from boundaries
- mathematical analysis [1,2,3] shows that commutation error is not negligible
- practice: no approach to incorporate commutation errors, they are simply neglected
- space-averaged Navier–Stokes equations in $(0,T] \times \mathbb{R}^d$

$$\begin{aligned} \partial_t \overline{\boldsymbol{u}} - 2\nu\nabla\cdot\mathbb{D}(\overline{\mathbf{u}}) + \nabla\cdot\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right) + \nabla\overline{p} &= \overline{\mathbf{f}} + \nabla\cdot\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T - \overline{\boldsymbol{u}\boldsymbol{u}^T}\right) \\ \nabla\cdot\overline{\mathbf{u}} &= 0 \end{aligned}$$

- main issue in LES : model $\nabla\cdot\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T - \overline{\boldsymbol{u}\boldsymbol{u}^T}\right)$ with $(\overline{\boldsymbol{u}}, \overline{p})$
  - many proposals

[1] Dunca, J., Layton: Contributions to Current Challenges in Mathematical Fluid Mechanics, Birkhäuser Verlag, 53 – 78, 2004

[2] Berselli, J.: Math. Methods Appl. Sci. 29, 1709 – 1719, 2006

[3] Berselli, Grisanti, J.: J. Comp. Appl. Math. 206, 1027 – 1045, 2007

- most popular LES model
- derivation based on physical understanding: Boussinesq hypothesis

Turbulent fluctuations are dissipative in the mean.

$$\Longrightarrow \quad \nabla \cdot \left(\overline{\boldsymbol{u}\boldsymbol{u}^T} - \overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T \right) \approx -\nabla \cdot (\nu_T \mathbb{D}(\overline{\boldsymbol{u}})) + \text{terms inc. in } \overline{p}$$

ν_T – eddy viscosity, turbulent viscosity

$$\nu_T = c_S \delta^2 \left\| \mathbb{D}(\overline{\boldsymbol{u}}) \right\|_{\mathrm{F}}$$

- no explicit filter (possible to derive for homogeneous isotropic turbulence)
- Smagorinsky parameter
  - δ should correspond to local mesh width (difficulty: anisotropic grids)
  - constant (often of order 0.01)
  - different proposal: functions in space and time (dynamical Smagorinsky model)

- weak equation posseses unique solution $\nabla \boldsymbol{w} \in L^3(0,T;L^3(\Omega))$ in 2d and 3d for large data and large time intervals [1]
  - more known than for Navier–Stokes equations (uniqueness in 3d)

Olga Alexandrovna Ladyzhenskaya (1922 – 2004)

[1] O.A. Ladyzhenskaya; Trudy Mat. Inst. Steklov. 102, 85–104, 1967

- weak equation posseses unique solution $\nabla \boldsymbol{w} \in L^3(0,T;L^3(\Omega))$ in 2d and 3d for large data and large time intervals [1]
  - more known than for Navier–Stokes equations (uniqueness in 3d)

Olga Alexandrovna Ladyzhenskaya (1922 – 2004)

- practical experience
  - introduces often too much viscosity $\Longrightarrow$ dynamical Smagorinsky model
  - smaller parameters at boundaries necessary, e.g., van Driest damping

[1] O.A. Ladyzhenskaya; Trudy Mat. Inst. Steklov. 102, 85–104, 1967

- analysis and modeling
  - commutation errors arise, partially analyzed, important near boundaries
  - Smagorinsky model with constant c_S well analyzed (existence, uniqueness of solution, finite element errors)
- practical application of LES models
  - many models proposed, used
  - Smagorinsky model (and variants) very popular
- literature
  - best reference [1]
  - more mathematical: [2,3]

[1] P. Sagaut; *Large eddy simulation for incompressible flows*, Springer, 2006

[2] L. Berselli, T. Iliescu, W.J. Layton; *Mathematics of large eddy simulation of turbulent flows*, Springer, 2006

[3] J.; *Finite Element Methods for Incompressible Flow Problems*, Springer Series in Computational Mathematics 51, 2016

- Variational Multiscale (VMS) methods
  - going back to [1,2]
  - main features
    - based on variational formulation of Navier–Stokes equations
    - scale separation defined by projections, different definition of large scales than in LES!
  - different realizations of VMS methods
  - survey on derivation, properties, mathematical results, computational experience in [3,4]

[1] Hughes; Comp. Meth. Appl. Mech. Engrg. 127, 387–401, 1995

[2] Guermond; M2AN 33, 1293–1316, 1999

[3] Ahmed, Chacón Rebollo, J., Rubino; Arch. Computat. Methods Engrg. 24, 115 – 164, 2017

[4] J.; *Finite Element Methods for Incompressible Flow Problems*, Springer Series in Computational Mathematics 51, 812 pages, 2016

- two-scale VMS method
- decomposition in resolved and small scales

$$\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u}', \quad p = \overline{p} + p'$$

gives decomposition of weak form of Navier–Stokes equations

$$\begin{array}{rcl} A\left(\boldsymbol{u}; \left(\overline{\boldsymbol{u}}, \overline{p}\right), \left(\overline{\boldsymbol{v}}, \overline{q}\right)\right) + A\left(\boldsymbol{u}; \left(\boldsymbol{u}', p'\right), \left(\overline{\boldsymbol{v}}, \overline{q}\right)\right) & = & F\left(\overline{\boldsymbol{v}}\right) \\ A\left(\boldsymbol{u}; \left(\overline{\boldsymbol{u}}, \overline{p}\right), \left(\boldsymbol{v}', q'\right)\right) + A\left(\boldsymbol{u}; \left(\boldsymbol{u}', p'\right), \left(\boldsymbol{v}', q'\right)\right) & = & F\left(\boldsymbol{v}'\right) \end{array}$$

- with notation

$$\boldsymbol{U} = \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} \boldsymbol{v} \\ q \end{pmatrix} \quad \text{and so on}$$

- decompose

$$A\left(\boldsymbol{u}; \boldsymbol{U}, \boldsymbol{V}\right) = A_{\mathrm{lin}}\left(\boldsymbol{U}, \boldsymbol{V}\right) + n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})$$

- rewrite small scale equation

$$A_{\boldsymbol{U}}\left(\boldsymbol{U}',\boldsymbol{V}'\right)+n\left(\boldsymbol{u}',\boldsymbol{u}',\boldsymbol{v}'\right)=-\left\langle \mathbf{Res}\left(\overline{\boldsymbol{U}}\right),\boldsymbol{V}'\right\rangle_{(V\times Q)',(V\times Q)}$$

with

$$\begin{aligned} A_{\boldsymbol{U}}\left(\boldsymbol{U}',\boldsymbol{V}'\right) &= A_{\text{lin}}\left(\boldsymbol{U}',\boldsymbol{V}'\right)+n\left(\boldsymbol{u}',\overline{\boldsymbol{u}},\boldsymbol{v}'\right)+n\left(\overline{\boldsymbol{u}},\boldsymbol{u}',\boldsymbol{v}'\right) \\ \left\langle \mathbf{Res}\left(\overline{\boldsymbol{U}}\right),\boldsymbol{V}'\right\rangle_{(V\times Q)',(V\times Q)} &= A_{\text{lin}}\left(\overline{\boldsymbol{U}},\boldsymbol{V}'\right)+n\left(\overline{\boldsymbol{u}},\overline{\boldsymbol{u}},\boldsymbol{v}'\right)-\left\langle \boldsymbol{f},\boldsymbol{v}'\right\rangle_{V',V} \end{aligned}$$

- interpretation: unresolved scales are a function of the residual of the resolved scales

$$\boldsymbol{U}'=\boldsymbol{F}_{\boldsymbol{U}}\left(-\mathbf{Res}\left(\overline{\boldsymbol{U}}\right)\right)\quad \text{or}\quad \boldsymbol{U}'=\boldsymbol{F}_{\boldsymbol{U}}\left(-\mathbf{Res}\left(\overline{\boldsymbol{U}}\right),\boldsymbol{u}'_{\text{old}}\right)$$

- goal of two-scale VMS method: find an approximation of $\boldsymbol{F}_{\boldsymbol{U}}$
  - usually NO physcial turbulence model involved, justification in [1]

[1] Guasch, Codina; Comp. Meth. Appl. Mech. Engrg. 261/262, 154–166, 2013

- three-scale VMS method
- decomposition of velocity and pressure
  - the large scales $(\overline{\boldsymbol{u}}, \overline{p})$
  - the small resolved scales $(\widehat{\boldsymbol{u}}, \widehat{p})$
  - the unresolved scales $(\boldsymbol{u}', p')$
- decomposition of equation

$$
\begin{aligned}
A\left(\boldsymbol{u};\left(\overline{\boldsymbol{u}},\overline{p}\right),\left(\overline{\boldsymbol{v}},\overline{q}\right)\right)+A\left(\boldsymbol{u};\left(\widehat{\boldsymbol{u}},\widehat{p}\right),\left(\overline{\boldsymbol{v}},\overline{q}\right)\right)+A\left(\boldsymbol{u};\left(\boldsymbol{u}',p'\right),\left(\overline{\boldsymbol{v}},\overline{q}\right)\right) &= F\left(\overline{\boldsymbol{v}}\right)\\
A\left(\boldsymbol{u};\left(\overline{\boldsymbol{u}},\overline{p}\right),\left(\widehat{\boldsymbol{v}},\widehat{q}\right)\right)+A\left(\boldsymbol{u};\left(\widehat{\boldsymbol{u}},\widehat{p}\right),\left(\widehat{\boldsymbol{v}},\widehat{q}\right)\right)+A\left(\boldsymbol{u};\left(\boldsymbol{u}',p'\right),\left(\widehat{\boldsymbol{v}},\widehat{q}\right)\right) &= F\left(\widehat{\boldsymbol{v}}\right)\\
A\left(\boldsymbol{u};\left(\overline{\boldsymbol{u}},\overline{p}\right),\left(\boldsymbol{v}',q'\right)\right)+A\left(\boldsymbol{u};\left(\widehat{\boldsymbol{u}},\widehat{p}\right),\left(\boldsymbol{v}',q'\right)\right)+A\left(\boldsymbol{u};\left(\boldsymbol{u}',p'\right),\left(\boldsymbol{v}',q'\right)\right) &= F\left(\boldsymbol{v}'\right)
\end{aligned}
$$

- neglect blue terms
  - unresolved scale test functions not available
  - direct impact of unresolved scales onto large scales can be neglected
- model orange term

- Find $(\overline{\boldsymbol{u}}, \widehat{\boldsymbol{u}}, \overline{p}, \widehat{p}) \in \overline{V} \times \widehat{V} \times \overline{Q} \times \widehat{Q}$ such that

$$\begin{aligned} A\left(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};\left(\overline{\boldsymbol{u}},\overline{p}\right),\left(\overline{\boldsymbol{v}},\overline{q}\right)\right)+A\left(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};\left(\widehat{\boldsymbol{u}},\widehat{p}\right),\left(\overline{\boldsymbol{v}},\overline{q}\right)\right) &= F\left(\overline{\boldsymbol{v}}\right), \\ A\left(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};\left(\overline{\boldsymbol{u}},\overline{p}\right),\left(\widehat{\boldsymbol{v}},\widehat{q}\right)\right)+A\left(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};\left(\widehat{\boldsymbol{u}},\widehat{p}\right),\left(\widehat{\boldsymbol{v}},\widehat{q}\right)\right) & \\ +T\left(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};\left(\overline{\boldsymbol{u}},\overline{p}\right),\left(\widehat{\boldsymbol{u}},\widehat{p}\right),\left(\widehat{\boldsymbol{v}},\widehat{q}\right)\right) &= F\left(\widehat{\boldsymbol{v}}\right) \end{aligned}$$

- model is usually physical based, like an eddy viscosity model (Smagorinsky model)

- approximation of $(\overline{\boldsymbol{u}}, \overline{p})$ denoted by $(\boldsymbol{w}^h, r^h)$

- proposed in [1]
- orthogonal spaces, with L^2 projection or elliptic projection

$$V \times Q = \left(\overline{V} \oplus V'\right) \times \left(\overline{Q} \oplus Q'\right),$$

- derivation based on perturbation series for $\varepsilon = \left\|\mathbf{Res}\left(\overline{\boldsymbol{U}}\right)\right\|_{(V\times Q)'}$, assumed to be small

$$\boldsymbol{U}' = \varepsilon \boldsymbol{U}_1' + \varepsilon^2 \boldsymbol{U}_2' + \ldots = \sum_{i=1}^{\infty} \varepsilon^i \boldsymbol{U}_i'$$

- inserting series in equation for small scales, ordering terms

$$\begin{aligned} A_{\boldsymbol{U}}\left(\boldsymbol{U}_1', \boldsymbol{V}'\right) &= -\left\langle \frac{\mathbf{Res}\left(\overline{\boldsymbol{U}}\right)}{\left\|\mathbf{Res}\left(\overline{\boldsymbol{U}}\right)\right\|_{(V\times Q)'}}, \boldsymbol{V}' \right\rangle_{(V\times Q)', (V\times Q)}, \\ A_{\boldsymbol{U}}\left(\boldsymbol{U}_i', \boldsymbol{V}'\right) &= -\sum_{j=1}^{i-1} n\left(\boldsymbol{u}_i', \boldsymbol{u}_j', \boldsymbol{v}'\right) \quad i \ge 2 \end{aligned}$$

[1] Bazilevs, Calo, Cottrell, Hughes, Reali, Scovazzi; Comput. Methods Appl. Mech. Engrg. 197, 173–201, 2007

- modeling steps
  - take only first term of series

$$\boldsymbol{U}' \approx \varepsilon \boldsymbol{U}_1' = \left\| \mathbf{Res}\left(\overline{\boldsymbol{U}}\right) \right\|_{(V\times Q)'} \boldsymbol{U}_1'$$

  - proposal from [1]: use linear approximation

$$\boldsymbol{U}_1' \approx -\boldsymbol{\delta} \frac{\mathbf{Res}\left(\overline{\boldsymbol{U}}\right)}{\left\| \mathbf{Res}\left(\overline{\boldsymbol{U}}\right) \right\|_{(V\times Q)'}},$$

$\boldsymbol{\delta}$ – stabilization parameter (tensor-valued)

[1] Bazilevs, Calo, Cottrell, Hughes, Reali, Scovazzi; Comput. Methods Appl. Mech. Engrg. 197, 173–201, 2007

- model of the small scales

$$\begin{aligned}\tilde{\boldsymbol{U}}' &= -\boldsymbol{\delta}\mathbf{Res}\left(\begin{pmatrix}\boldsymbol{w}^h\\ r^h\end{pmatrix}\right)\\ &= -\begin{pmatrix}\boldsymbol{\delta}_{\mathrm{m}}\left(\partial_t\boldsymbol{w}^h - \nu\Delta\boldsymbol{w}^h + \left(\boldsymbol{w}^h\cdot\nabla\right)\boldsymbol{w}^h + \nabla r^h - \boldsymbol{f}\right)\\ \mu\left(\nabla\cdot\boldsymbol{w}^h\right)\end{pmatrix} = -\begin{pmatrix}\mathbf{res}_{\mathrm{m}}^h\\ \mathrm{res}_{\mathrm{c}}^h\end{pmatrix}\end{aligned}$$

- resulting method: Find $\boldsymbol{w}^h \ : \ (0,T] \to V^h,\ r^h \ : \ (0,T] \to Q^h$ satisfying

$$\begin{aligned}&\left(\partial_t\boldsymbol{w}^h,\boldsymbol{v}^h\right) + \left(2\nu\mathbb{D}\left(\boldsymbol{w}^h\right),\mathbb{D}\left(\boldsymbol{v}^h\right)\right) + n\left(\boldsymbol{w}^h,\boldsymbol{w}^h,\boldsymbol{v}^h\right) + \left(\nabla\cdot\boldsymbol{w}^h,q^h\right)\\ &\quad - \left(\nabla\cdot\boldsymbol{v}^h,r^h\right) + \left(\mathrm{res}_{\mathrm{c}}^h,\nabla\cdot\boldsymbol{v}^h\right) + \left(\mathbf{res}_{\mathrm{m}}^h,\nabla q^h\right) - n\left(\mathbf{res}_{\mathrm{m}}^h,\boldsymbol{w}^h,\boldsymbol{v}^h\right)\\ &\quad - n\left(\boldsymbol{w}^h,\mathbf{res}_{\mathrm{m}}^h,\boldsymbol{v}^h\right) + n\left(\mathbf{res}_{\mathrm{m}}^h,\mathbf{res}_{\mathrm{m}}^h,\boldsymbol{v}^h\right) = \langle\boldsymbol{f},\boldsymbol{v}^h\rangle_{V',V}\end{aligned}$$

for all $\left(\boldsymbol{v}^h,q^h\right)\in V^h\times Q^h$

- the additional terms
  - grad-div stabilization
  - SUPG terms
  - model for second cross term and subgrid scale term
- proposal of stabilization parameter in [1] gives for uniform meshes and convection-dominated regime $\delta_{\mathrm{m}} \sim h$ and $\mu \sim h$
  - optimal choice for SUPG/PSPG/grad-div method for Oseen equations and equal-order interpolations
- numerical studies in the literature only for equal-order interpolations

[1] Bazilevs, Calo, Cottrell, Hughes, Reali, Scovazzi; Comput. Methods Appl. Mech. Engrg. 197, 173–201, 2007

- two-scale method, proposed in [1]
- large scale spaces: finite element spaces $V^h \times Q^h$
- small scale spaces: $\tilde{V}' \times \tilde{Q}'$ such that

$$V = V^h \oplus \tilde{V}' \quad \text{and} \quad Q = Q^h \oplus \tilde{Q}'$$

- small scale equation with integration by parts and smooth solution

$$\begin{aligned} \langle \partial_t \boldsymbol{u}' - \nu \Delta \boldsymbol{u}' + (\boldsymbol{u}' \cdot \nabla)\, \overline{\boldsymbol{u}} + (\overline{\boldsymbol{u}} \cdot \nabla)\, \boldsymbol{u}' + (\boldsymbol{u}' \cdot \nabla)\, \boldsymbol{u}' + \nabla p', \boldsymbol{v}' \rangle_{V',V} + (\nabla \cdot \boldsymbol{u}', q') \\ = \quad - \langle \partial_t \overline{\boldsymbol{u}} - \nu \Delta \overline{\boldsymbol{u}} + (\overline{\boldsymbol{u}} \cdot \nabla)\, \overline{\boldsymbol{u}} + \nabla \overline{p} - \boldsymbol{f}, \boldsymbol{v}' \rangle_{V',V} - (\nabla \cdot \overline{\boldsymbol{u}}, q') \end{aligned}$$

[1] Codina; Comp. Meth. Appl. Mech. Engrg. 191, 4295–4321, 2002

- first idea: do not neglect temporal derivative of small scales

$$\partial_t \boldsymbol{u}' \approx \vartheta \frac{\boldsymbol{u}' - \boldsymbol{u}'_{\text{old}}}{\Delta t},$$

  ϑ depends on temporal discretization
- linear ansatz for small scales

$$\boldsymbol{U}' = -\boldsymbol{\delta} \left(\mathbf{Res} \left(\overline{\boldsymbol{U}}, \boldsymbol{u}'_{\text{old}} \right) + \boldsymbol{V}_{\text{orth}} \right)$$

- second idea: subscales should be $L^2(\Omega)$ orthogonal to finite element space $\Longrightarrow$

$$\boldsymbol{\delta V}_{\text{orth}} = -P^h_{L^2} \left(\boldsymbol{\delta} \mathbf{Res} \left(\overline{\boldsymbol{U}}, \boldsymbol{u}'_{\text{old}} \right) \right)$$

  and

$$\boldsymbol{U}' = -\left(I - P^h_{L^2} \right) \left(\boldsymbol{\delta} \mathbf{Res} \left(\overline{\boldsymbol{U}}, \boldsymbol{u}'_{\text{old}} \right) \right)$$

- some simplifications applied for practical reasons

- find $(\boldsymbol{w}^h, r^h)$ such that

$$
\begin{aligned}
& \left(\partial_t \boldsymbol{w}^h, \boldsymbol{v}^h\right) + \nu\left(\nabla \boldsymbol{w}^h, \nabla \boldsymbol{v}^h\right) + n\left(\boldsymbol{w}^h + \boldsymbol{w}', \boldsymbol{w}^h, \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, r^h\right) \\
& \qquad + \left(\nabla \cdot \boldsymbol{w}^h, r^h\right) + \sum_{K \in \mathcal{T}^h} \Bigg(\left(I - P_{L^2}^h\right)\left(\left(\left(\boldsymbol{w}^h + \boldsymbol{w}'\right) \cdot \nabla\right) \boldsymbol{w}^h + \nabla r^h\right), \\
& \qquad\qquad \delta_{\mathrm{m},K}\left(\left(\left(\boldsymbol{w}^h + \boldsymbol{w}'\right) \cdot \nabla\right) \boldsymbol{v}^h + \nabla q^h\right) \Bigg)_K \\
& \qquad + \sum_{K \in \mathcal{T}^h} \left(\left(I - P_{L^2}^h\right)\left(\nabla \cdot \boldsymbol{w}^h\right), \mu_K \nabla \cdot \boldsymbol{v}^h\right)_K \\
= & \quad \left(\boldsymbol{f}, \boldsymbol{v}^h\right) + \sum_{K \in \mathcal{T}^h} \left(\left(I - P_{L^2}^h\right)\boldsymbol{f}, \delta_{\mathrm{m},K}\left(\left(\left(\boldsymbol{w}^h + \boldsymbol{w}'\right) \cdot \nabla\right) \boldsymbol{v}^h + \nabla q^h\right)\right)_K \\
& \qquad + \frac{\vartheta}{\Delta t} \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{w}'_{\mathrm{old}}, \delta_{\mathrm{m},K}\left(\left(\left(\boldsymbol{w}^h + \boldsymbol{w}'\right) \cdot \nabla\right) \boldsymbol{v}^h + \nabla q^h\right)\right)_K
\end{aligned}
$$

- global projection
- advection velocity

- equation for subscales

$$\boldsymbol{w}'|_K = \delta_{\mathrm{m},K} \left(\vartheta \frac{\boldsymbol{w}'_{\mathrm{old}}}{\Delta t} - \left(I - P_{L^2}^h\right) \left(\left(\left(\boldsymbol{w}^h + \boldsymbol{w}' \right) \cdot \nabla \right) \boldsymbol{w}^h + \nabla r^h - \boldsymbol{f} \right) \right) \Big|_K$$

- variations/refinements/extensions of this prototype method exist
- stabilization parameters in [1] for equal-order finite elements
- numerical studies in literature only for equal-order finite elements
- $\boldsymbol{w}'_{\mathrm{old}} = \mathbf{0}$: static or quasi-static subscales
- $\boldsymbol{V}_{\mathrm{orth}} = \mathbf{0}$: algebraic subgrid scale (ASGS) VMS method
- ASGS with static subscales: same derivation principles as for two-scale residual-based VMS method from [2] (but different final method)
- backscatter of energy only subscales are time-dependent [3,4]

[1] Codina; Comp. Meth. Appl. Mech. Engrg. 191, 4295–4321, 2002

[2] Bazilevs, Calo, Cottrell, Hughes, Reali, Scovazzi; Comput. Methods Appl. Mech. Engrg. 197, 173–201, 2007

[3] Principe, Codina, Henke; Comp. Meth. Appl. Mech. Engrg. 199, 791–801, 2010

[4] Codina, Principe, Badia; Lect. Notes Appl. Comput. Mech. 55, 75–93, 2011

- three-scale method
  - AVM^3 proposed in [1,2]
  - AVM^4 proposed in [3]
- definition of the separation of the resolved scales
  - uses ideas from algebraic multigrid (AMG) methods
  - does not need another finite element space or another grid
  - definition of the large scales

$$S_h^{3h} \,:\, V^h \to V^h, \quad \boldsymbol{u}^{3h} = P_{3h}^h R_h^{3h} \boldsymbol{u}^h$$

R_h^{3h} – restriction, plain aggregation, $P_{3h}^h = \left(R_{3h}^h\right)^T$

  - scale separation, $\widehat{\boldsymbol{u}}^h$ – small resolved scales

$$\boldsymbol{u}^h = \boldsymbol{u}^{3h} + \widehat{\boldsymbol{u}}^h \quad \Longleftrightarrow \quad \widehat{\boldsymbol{u}}^h = \boldsymbol{u}^h - \boldsymbol{u}^{3h}$$

[1] Gravemeier, Gee, Wall; Comp. Meth. Appl. Mech. Engrg. 198, 3821–3835, 2009

[2] Gravemeier, Gee, Kronbichler, Wall; Comp. Meth. Appl. Mech. Engrg. 199, 853–864, 2010

[3] Rasthofer, Gravemeier; J. Comput. Phys. 234, 79–107, 2013

- AVM^3
  - start with two-scale decomposition

$$\boldsymbol{u} = \boldsymbol{u}^h + \boldsymbol{u}', \quad p = p^h + p'$$

  - neglect equation for unresolved test function

$$\begin{aligned} &\left(\partial_t \boldsymbol{u}^h, \boldsymbol{v}^h\right) + \left(2\nu\mathbb{D}\left(\boldsymbol{u}^h\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) + \left(\left(\boldsymbol{u}^h \cdot \nabla\right)\boldsymbol{u}^h, \boldsymbol{v}^h\right) + \left(\nabla \cdot \boldsymbol{u}^h, q^h\right) \\ &\qquad - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) \\ &= \quad \left(\boldsymbol{f}, \boldsymbol{v}^h\right) - \Big[\left(\partial_t \boldsymbol{u}', \boldsymbol{v}^h\right) + \left(2\nu\mathbb{D}\left(\boldsymbol{u}'\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) + \left(\left(\boldsymbol{u}^h \cdot \nabla\right)\boldsymbol{u}', \boldsymbol{v}^h\right) \\ &\qquad + \left(\left(\boldsymbol{u}' \cdot \nabla\right)\boldsymbol{u}^h, \boldsymbol{v}^h\right) + \left(\left(\boldsymbol{u}' \cdot \nabla\right)\boldsymbol{u}', \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p'\right) \Big] - \left(\nabla \cdot \boldsymbol{u}', q^h\right) \end{aligned}$$

  - split test function $\boldsymbol{v}^h = \boldsymbol{v}^{3h} + \widehat{\boldsymbol{v}}^h$ in blue term
  - apply assumption for a three-scale VMS method to blue term
    - direct impact of unresolved scales on large scales is neglected
    - model direct impact of unresolved scales on small resolved scales

- AVM^3 (cont.)
  - in [1]: Smagorinsky model

$$\nabla \cdot \left(C_S h^2 \left\| \mathbb{D}\left(\widehat{\boldsymbol{u}}^h\right) \right\|_{\mathrm{F}} \mathbb{D}\left(\widehat{\boldsymbol{u}}^h\right)\right) = \nabla \cdot \left(\nu_{\mathrm{T}} \left(\widehat{\boldsymbol{u}}^h\right) \mathbb{D}\left(\widehat{\boldsymbol{u}}^h\right)\right)$$

  - realizations of AVM^3 only for Q_1/Q_1
    - in practice added PSPG-type stabilization
  - short form

$$\begin{aligned} A\left(\boldsymbol{w}^h; \left(\boldsymbol{w}^h, r^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) &+ \text{PSPG-type stabilization} \\ &+ \left(\nu_T\left(\widehat{\boldsymbol{w}}^h\right)\mathbb{D}\left(\widehat{\boldsymbol{w}}^h\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) \quad = \quad F\left(\boldsymbol{v}^h\right) \end{aligned}$$

- AVM^4 – algebraic multiscale-multigrid-multifractal method
  - uses so-called multifractal model of $\boldsymbol{u}'$ instead of eddy viscosity model

[1] Gravemeier, Gee, Kronbichler, Wall; Comp. Meth. Appl. Mech. Engrg. 199, 853–864, 2010

- three-scale method
  - based on ideas from [1]
  - complete development and realization in [2]
- large scale space L^H – space of symmetric tensor-valued functions
- method

$$\begin{aligned}\left(\partial_t \boldsymbol{w}^h, \boldsymbol{v}^h\right) + \left(2\nu\mathbb{D}\left(\boldsymbol{w}^h\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) + n(\boldsymbol{w}^h, \boldsymbol{w}^h, \boldsymbol{v}^h) & \\ - \left(\nabla\cdot\boldsymbol{v}^h, r^h\right) + \left(\nu_{\mathrm{T}}\left(\mathbb{D}\left(\boldsymbol{w}^h\right) - \mathbb{G}^H\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) &= \langle \boldsymbol{f}, \boldsymbol{v}^h\rangle_{V',V} \\ \left(\nabla\cdot\boldsymbol{w}^h, q^h\right) &= 0 \\ \left(\mathbb{D}\left(\boldsymbol{w}^h\right) - \mathbb{G}^H, \mathbb{L}^H\right) &= 0\end{aligned}$$

  - definition of small resolved scales by projection

[1] Layton; Appl. Math. Comput. 133, 147–157, 2002

[2] J., Kaya; SIAM J. Sci. Comp. 26, 1485–1503, 2005

- short form

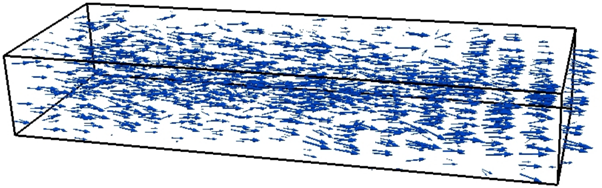
$$A\left(\boldsymbol{w}^h;\left(\boldsymbol{w}^h,r^h\right),\left(\boldsymbol{v}^h,q^h\right)\right)+\left(\nu_{\mathrm{T}}\left(\mathbb{D}\left(\boldsymbol{w}^h\right)-\mathbb{G}^H\right),\mathbb{D}\left(\boldsymbol{v}^h\right)\right)=F\left(\boldsymbol{v}^h\right)$$

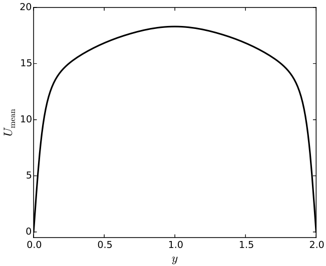
  - similar structure as for AVM^3, AVM^4
- generally Smagorinsky type models used for ν_{T}
- choice of L^H: discontinuous space on the same grid $(P_0, P_1^{\mathrm{disc}}$, mixed in an adaptive algorithm)

- local projection stabilization (LPS) methods
  - can be considered as two-scale VMS methods [1]
  - structure is obtained by replacing in VMS method with time-dependent orthogonal subscales the global $L^2(\Omega)$ projection by a local projection
- three-scale bubble VMS methods
  - represent small resolved scales with bubble functions

[1] Braack, Burman; SIAM J. Numer. Anal. 43, 2544–2566, 2006

- turbulent channel flow at $\mathrm{Re}_\tau = 180$ (friction Reynolds number)

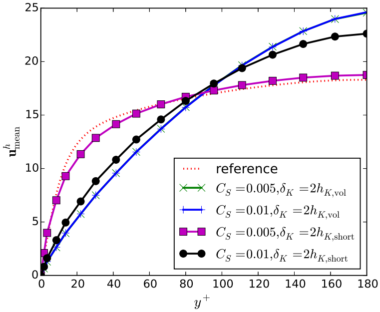


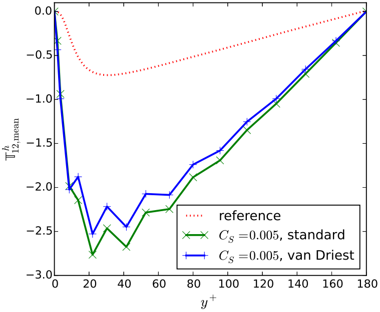


mean velocity profile

- very coarse grid: $8 \times 16 \times 8$ cells, finer towards the walls

- Smagorinsky LES model



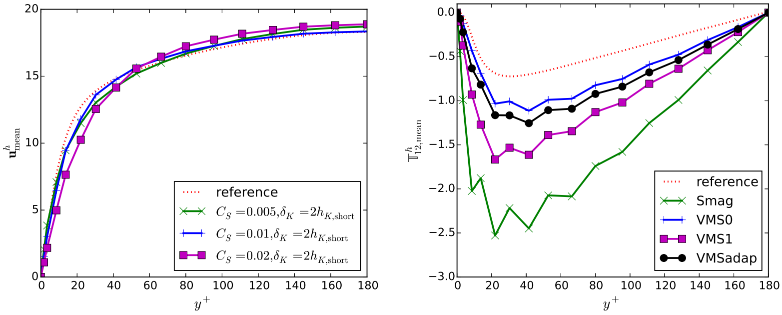


- second order statistics

$$\mathbb{T}^h_{12,\mathrm{mean}} = \langle\langle u_1^h u_2^h\rangle_s\rangle_t - \langle\langle u_1^h\rangle_s\rangle_t\langle\langle u_2^h\rangle_s\rangle_t$$

- very sensitive to the choice of the Smagorinsky parameter

- projection-based VMS models



  - left: VMS model with different parameters
  - right: comparison with Smagorinsky LES model for second order statistics
- VMS model more accurate
  - consistent statement in literature for all VMS models

- simulation of turbulent flows requires turbulence modeling
  - situation will not change in foreseeable future
- much progress in past two decades
  - LES approaches developed further
  - VMS methods introduced
- VMS models
  - distinguish two-scale and three-scale VMS methods
  - different realizations
    - usually only in the group of the developer of the method
  - ongoing project: implementation of various VMS methods in in-house code ParMooN (Parallel Mathematics and object-oriented Numerics) [1]
- more details in [2] and [3]

[1] Wilbrandt, Bartsch, et al.; Comput. Math. Appl. 74, 74–88, 2017

[2] Ahmed, Chacón Rebollo, J., Rubino; Arch. Computat. Methods Engrg. 24, 115 – 164, 2017

[3] J.; *Finite Element Methods for Incompressible Flow Problems*, Springer Series in Computational Mathematics 51, 812 pages, 2016

Thank you for your attention !

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