

Getting high (contrast) with LSD (Localized Spectrum Decomposition)

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Finite Element Method

Mixed LOD - a primal hybrid formulation on steroids

LSD - localized spectral decomposition (for high-contrast PDEs)

Conclusions

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Continuous problem with heterogeneous coefficients

Find $u \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \mathcal{A} \nabla u \cdot \nabla v \, dx = \int_{\Omega} gv \, dx \quad \text{for all } v \in H_0^1(\Omega)$$

where

- ▶ $\Omega \subset \mathbb{R}^d$ polygonal
- ▶ $g \in L^2(\Omega)$ piece-wise constant (for simplicity)
- ▶ $\mathcal{A} \in [L^\infty(\Omega)]^{d \times d}_{\text{sym}}$ uniformly coercive

Hurdles: low regularity, high-contrast and multiscale

Let

$$a(u, v) \stackrel{\text{def}}{=} \int_{\Omega} \mathcal{A} \nabla u \cdot \nabla v \, dx, \quad (g, v) \stackrel{\text{def}}{=} \int_{\Omega} gv \, dx$$

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Messages

Two main points of the talk

1. “smart” space decomposition (not ours)
 - ▶ allow local or “quasi-local” static condensation
2. spectral decomposition of spaces
 - ▶ based on local eigenvalue problems
 - ▶ make the scheme robust w.r.t. high contrast

Finite Elements

- ▶ \mathcal{T}_H nice partition of Ω
- ▶ $V_H = \{v \in H_0^1(\Omega) : v \text{ piece-wise linear}\}$
- ▶ $u_H \in V_H$ such that

$$a(u_H, v_H) = (g, v_H) \quad \text{for all } v_H \in V_H$$

- ▶ error analysis:

$$\|u - u_H\|_{H^1(\Omega)} \leq C H |u|_{H^2(\Omega)}$$

- ▶ $|u|_{H^2(\Omega)}$ and C not good for multiscale problems

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High Contrast

Consider

$$u^\epsilon = \arg \min_{v \in H_0^1(0,1)} \int_0^1 \frac{1}{2} |a(x/\epsilon)v'(x)|^2 - v(x) dx.$$

where

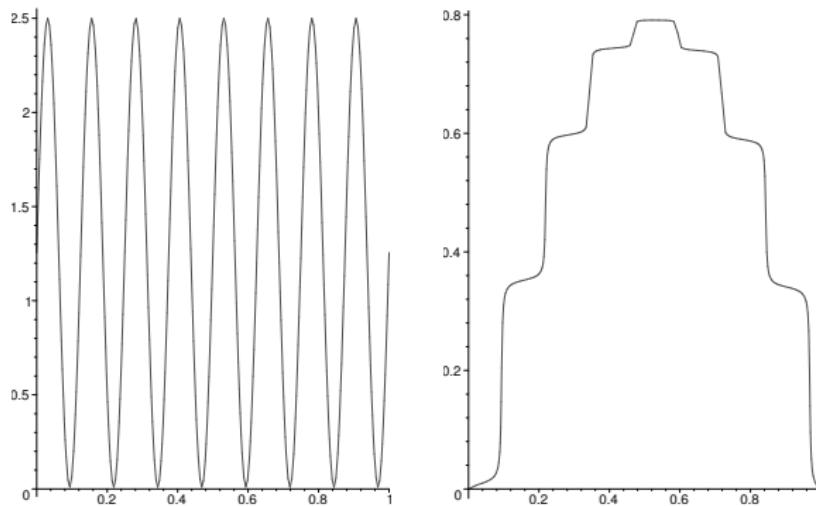


Figure: $a(\cdot/\epsilon)$ and exact solution for $\epsilon = 1/8$ (contrast=250)

High Contrast

Homogenization and finite elements fail:

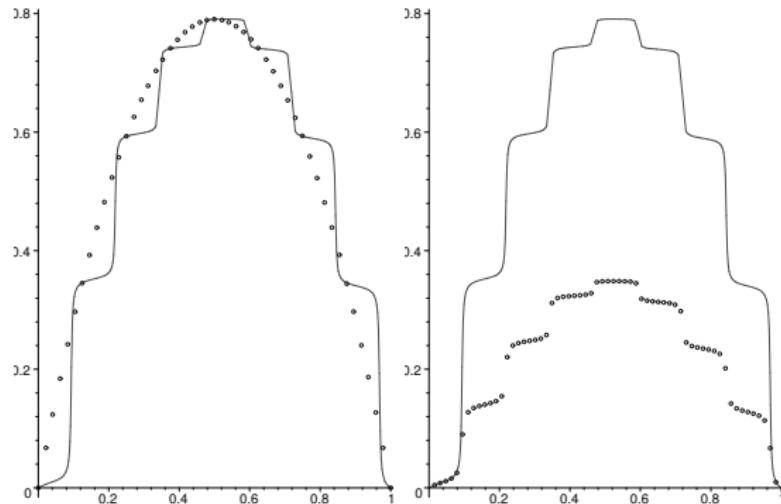


Figure: Comparison between the exact, homogenized solutions and finite element approximation for $h = 1/64$, for $\epsilon = 1/8$.

High Contrast

Even for homogeneous media, finite elements fail under high contrast:

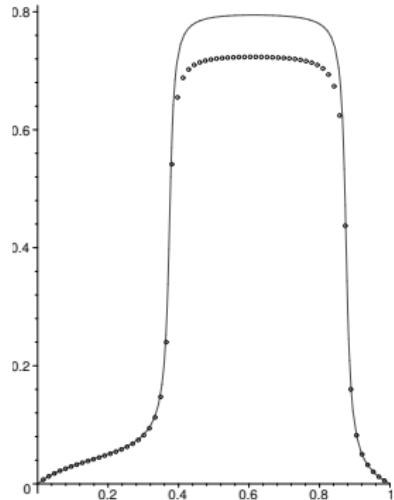


Figure: Exact and finite element solutions for $\epsilon = 1/2$ and $h = 1/64$

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Primal hybrid formulation

Spaces

- ▶ $H^1(\mathcal{T}_H) = \{v \in L^2(\Omega) : v|_\tau \in H^1(\tau), \tau \in \mathcal{T}_H\}$
- ▶ $\Lambda(\mathcal{T}_H) = \{\prod_{\tau \in \mathcal{T}_H} \boldsymbol{\sigma} \cdot \mathbf{n}^\tau|_{\partial\tau} : \boldsymbol{\sigma} \in H(\text{div}; \Omega)\}$

Hybrid formulation: $u \in H^1(\mathcal{T}_H)$, $\lambda \in \Lambda(\mathcal{T}_H)$ solve

$$\sum_{K \in \mathcal{T}_h} \int_K \mathcal{A} \nabla u \cdot \nabla v \, d\mathbf{x} + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \lambda v \, d\mathbf{x} = \int_{\Omega} g v \, d\mathbf{x} \quad \forall v \in H^1(\mathcal{T}_H)$$
$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu u \, d\mathbf{x} = 0 \quad \forall \mu \in \Lambda(\mathcal{T}_H)$$

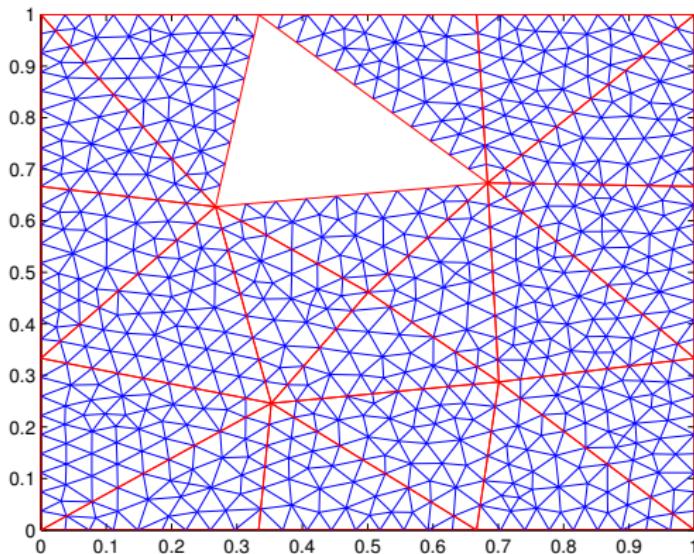
Comments

- ▶ first eqtn: equilibrium and $\lambda = -(\mathcal{A} \nabla u) \mathbf{n}^K$ on ∂K
- ▶ second eqtn: continuity and Dirichlet boundary condition
- ▶ Pian & Tong, 69'-71'; Raviart & Thomas 77'

Can be a large problem...

practical implementation:

- ▶ introduce sub-mesh \mathcal{T}_h
- ▶ \mathcal{F}_h be a partition of the faces of elements in \mathcal{T}_H
- ▶ $\Lambda_h = \{\mu_h \in \Lambda(\mathcal{T}_H) : \mu_h|_{F_h} \text{ is const on faces } F_h \in \mathcal{F}_h\}$



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- ▶ \mathcal{F}_h be a partition of the faces of elements in \mathcal{T}_H
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Assume that

$$|u - u_h|_{H_A^1(\mathcal{T}_H)} = |\lambda - \lambda_h|_{\Lambda(\mathcal{T}_H)} \leq \mathcal{H},$$

for some desired precision \mathcal{H}

“Classical” way out: hybridization based on fluxes

- ▶ FETI, MHM, HDG, M³FEM, MRCM...

Hybridization example

Decompose (FETI & MHM): $H^1(\mathcal{T}_H) = \mathbb{P}^0(\mathcal{T}_H) \oplus \tilde{H}^1(\mathcal{T}_H)$

- ▶ $\mathbb{P}^0(\mathcal{T}_H)$ - constant by parts (dim = # coarse elements)
- ▶ Λ_h - flux space (dim $\sim H^{-d}h^{-d+1}$)

Then $u_h = u_h^0 + T\lambda_h$ where

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \lambda_h T \mu_h d\mathbf{x} + \sum_{K \in \mathcal{T}_h} \int_{\partial K} u_h^0 \mu_h d\mathbf{x} = 0 \quad \forall \mu_h \in \Lambda_h$$
$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \lambda_h v_h^0 d\mathbf{x} = \int_{\Omega} g v_h^0 d\mathbf{x} \quad \forall v \in \mathbb{P}^0(\mathcal{T}_H)$$

where T is local \mathcal{A} -harmonic extension (with Neumann bc)

Not good enough yet:

- ▶ system size \sim flux space. Still large...
- ▶ goal: final system depending *only* on H

Time for digression!

Major sources:

- ▶ VMS (Hughes '95)
- ▶ LOD (Målqvist '11, Målqvist & Peterseim '14)
- ▶ “Mixed LOD” (Målqvist, Henning & Hellman '16)
- ▶ Among others

Decompose the fluxes!

Decompose: $\Lambda_h = \Lambda^{\text{const}} \oplus \tilde{\Lambda}_h^f$

- ▶ Λ^{const} - constant on edges (dim = # edges)
- ▶ $\tilde{\Lambda}_h^f$ - zero average on edges (dim $\sim h^{-1}$ # edges)
- ▶ $\lambda_h = \lambda_h^{\text{const}} + \tilde{\lambda}_h^f$

Then $u_h = u_h^0 + T\lambda_h$ and $\lambda_h = \lambda_h^{\text{const}} + \tilde{\lambda}_h^f$

$$(\lambda_h^{\text{const}}, v^0)_{\partial\mathcal{T}_H} = -(g, v^0)_{\mathcal{T}_H}$$

$$(\tilde{\mu}_h^f, T\lambda_h^{\text{const}} + T\tilde{\lambda}_h^f)_{\partial\mathcal{T}_H} = 0$$

$$(\mu^{\text{const}}, T\lambda_h^{\text{const}} + T\tilde{\lambda}_h^f)_{\partial\mathcal{T}_H} + (\mu^{\text{const}}, u_h^0)_{\partial\mathcal{T}_H} = 0$$

One large system: solve $\tilde{\lambda}_h^f$ w.r.t. λ_h^{const}

Static Condensation

Let $PT : \Lambda_h \rightarrow \tilde{\Lambda}_h^f$ such that, given $\lambda \in \Lambda_h$,

$$(\tilde{\mu}_h^f, TPT\lambda)_{\partial\mathcal{T}_H} = (\tilde{\mu}_h^f, T\lambda)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^f \in \tilde{\Lambda}_h^f$$

Then $\tilde{\lambda}_h^f = -PT\lambda_h^{\text{const}}$, and it's enough to solve:

$$\begin{aligned} ((I - PT)\mu_h^{\text{const}}, T(I - PT)\lambda_h^{\text{const}})_{\partial\mathcal{T}_H} + (\mu_h^{\text{const}}, u_h^0)_{\partial\mathcal{T}_H} &= 0 \\ (\lambda_h^{\text{const}}, v^0)_{\partial\mathcal{T}_H} &= -(g, v^0)_{\mathcal{T}_H} \end{aligned}$$

Good:

- ▶ problem size $\sim \# \text{ coarse space}$
- ▶ “elliptic system”

Not good:

- ▶ P is not local
- ▶ solving P is as hard as the whole problem

Localize

Let $\lambda \in \Lambda_h$ with local support

- ▶ then $PT\lambda$ decays exponentially!!!
- ▶ solve $PT\lambda$ in a patch of j elements around support of λ (we call it $P^j T\lambda$)
- ▶ replace P by P^j in the final system

$$\begin{aligned} ((I - P^j T)\mu_h^{\text{const}}, T(I - P^j T)\lambda_h^{\text{const}})_{\partial\mathcal{T}_H} + (\mu_h^{\text{const}}, u_h^0)_{\partial\mathcal{T}_H} &= 0 \\ (\lambda_h^{\text{const}}, v^0)_{\partial\mathcal{T}_H} &= -(g, v^0)_{\mathcal{T}_H} \end{aligned}$$

Only good news

- ▶ problems size $\sim \#$ coarse space
- ▶ “elliptic system”

Energy decay

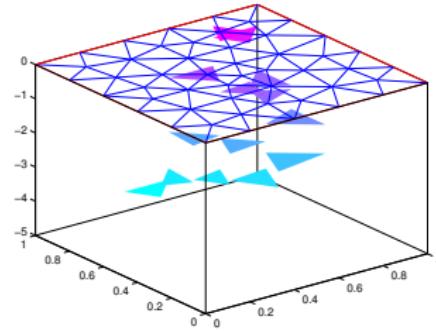
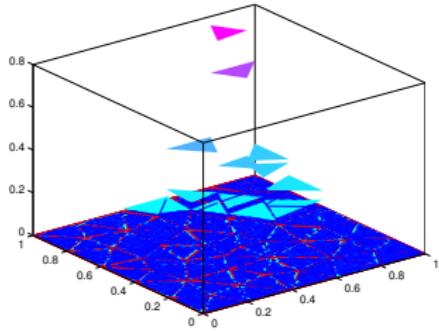
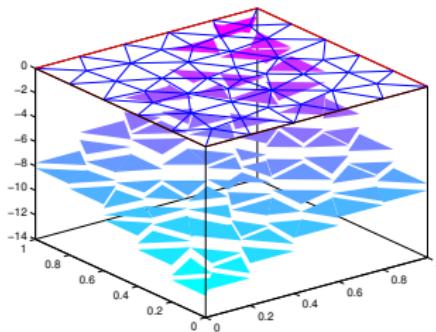
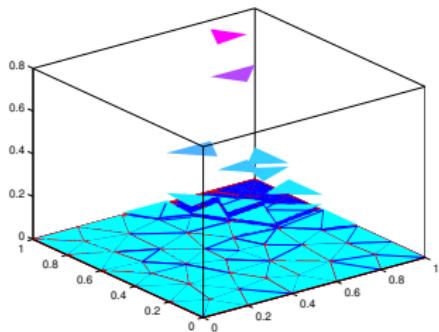


Figure: Energy decay of $PT\lambda$ (top) and $P^j T \lambda$ (bottom); real energy (left) and log of energy (right)

Energy decay

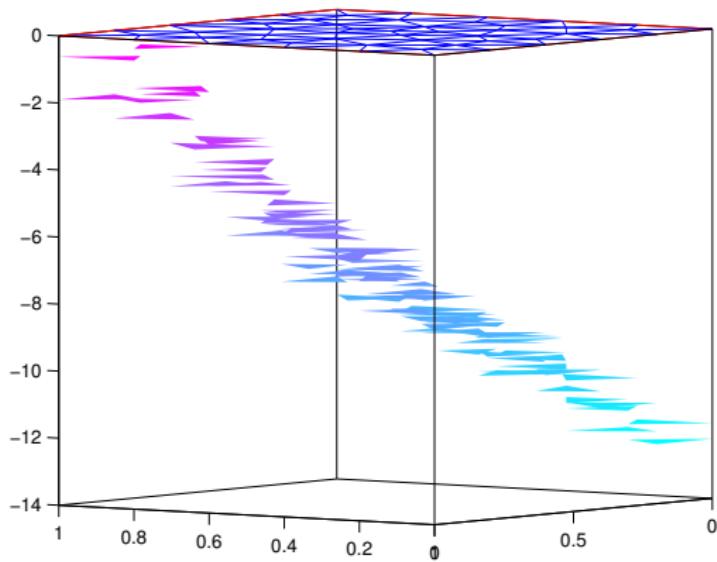


Figure: Log plot of energy of $PT\lambda$. Log of decay should be linear

Error estimate

In energy norm:

$$|u - u_h^j|_{H_A^1(\mathcal{T}_H)} \leq \mathcal{H} + c j^d e^{-\frac{j-2}{c\beta_{H/h}}} \beta_{H/h} \|g\|_{L^2(\Omega)}$$

where $\beta_{H/h} = 1 + \log(H/h)$

Comments

- ▶ two parts: \mathcal{H} (accuracy) and exponential term
- ▶ if $h \rightarrow 0$ then $\beta_{H/h}$ grows
- ▶ if patch size j grows, error $\rightarrow \mathcal{H}$
- ▶ needs minimum regularity, i.e., $u \in H^1(\Omega)$
- ▶ **c depends on the contrast**

Examples

Consider $\Omega = [0, 1]^2$, $g = 1$, and the mesh

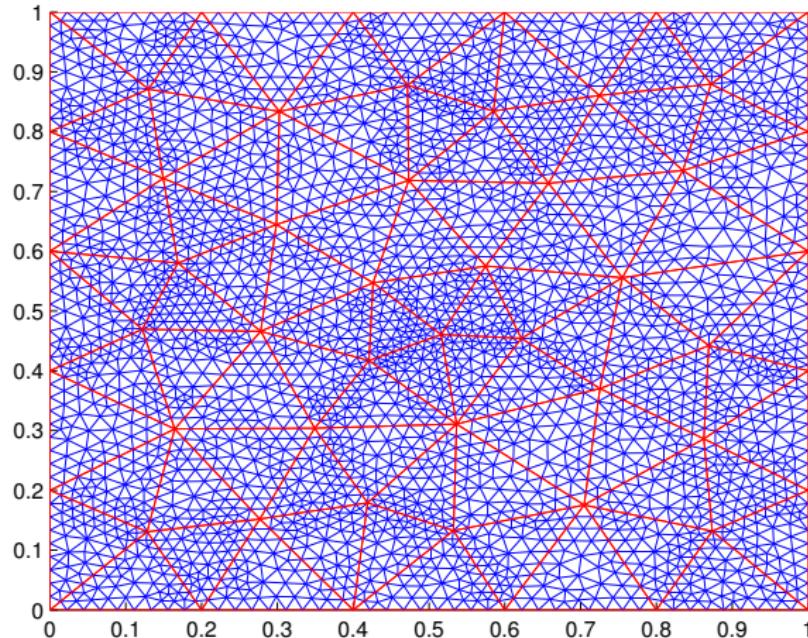


Figure: Mesh with $H = 0.2$ and $h \sim H/8$

High-contrast case

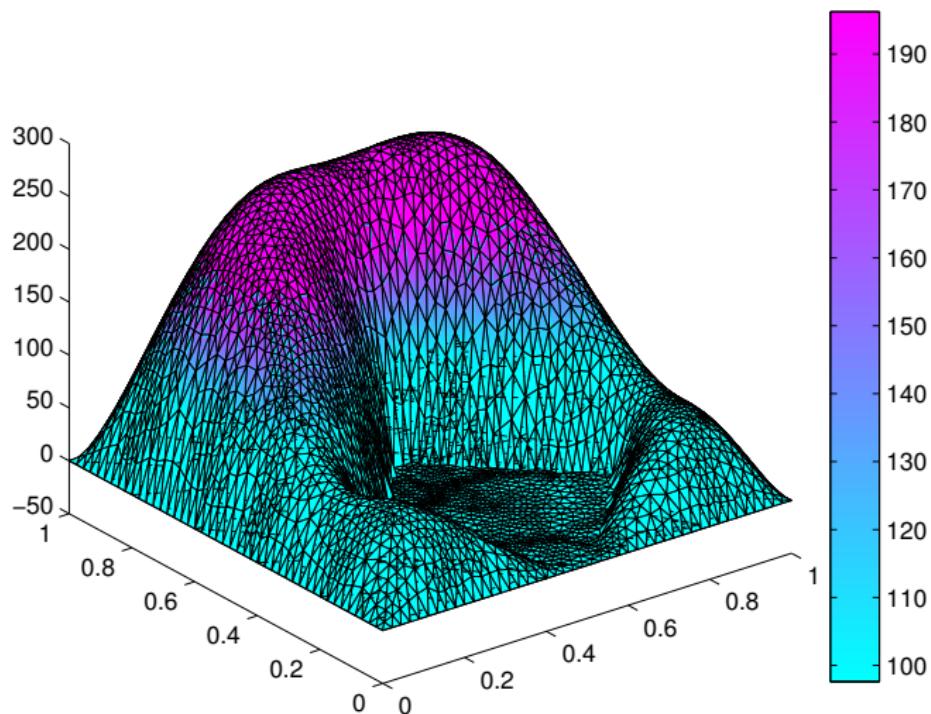


Figure: "Exact" solution u_h

High-contrast case

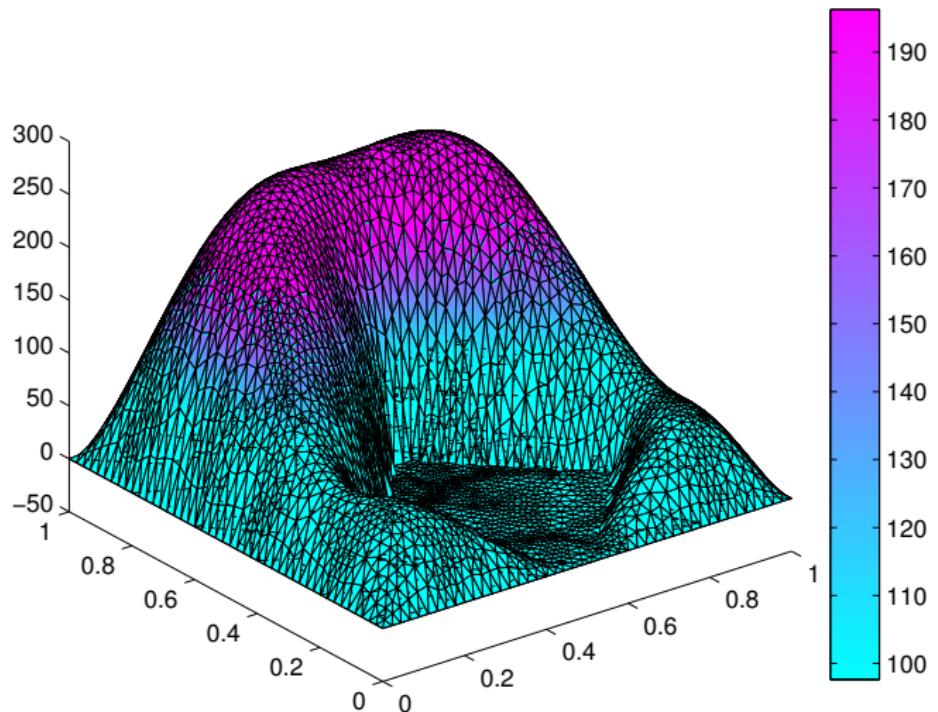


Figure: Approximate solution with patch of four elements

High-contrast case

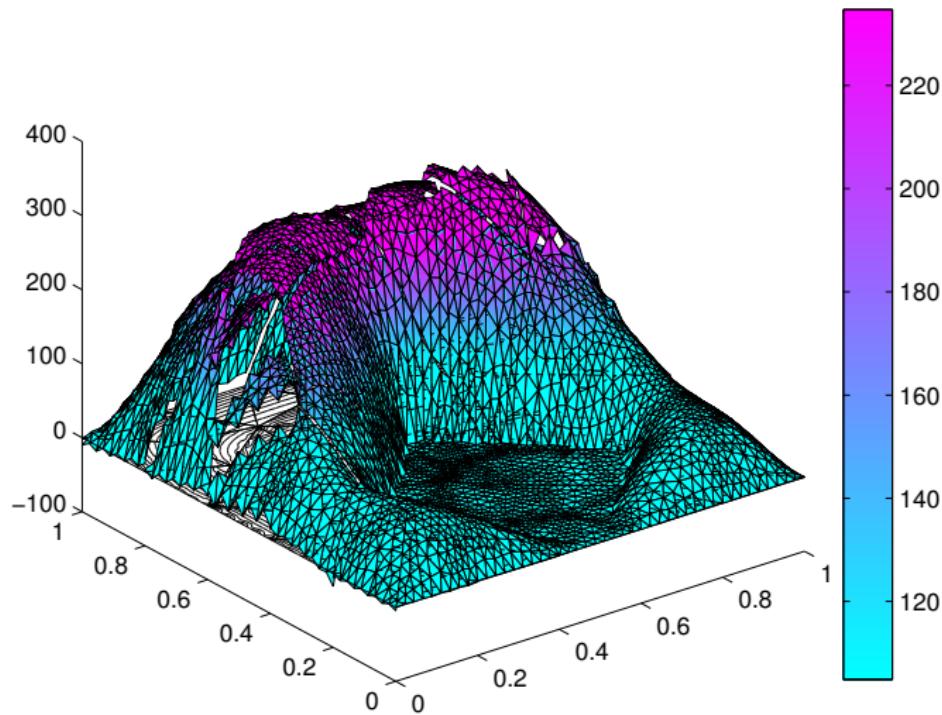


Figure: Approximate solution with patch of three elements

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Spectral Decomposition

- ▶ Fix a face $F \in \partial\mathcal{T}_H$, shared by elements τ and τ'
- ▶ For the element τ , let $F_\tau^c = \partial\tau \setminus F$.
- ▶ Let $T : \Lambda_h \rightarrow \Lambda'_h$ be Neumann-to-Dirichlet operator on $\partial\tau$:

$$T\lambda_h^\tau = \begin{pmatrix} T_{FF}^\tau & T_{F^cF}^\tau \\ T_{FF^c}^\tau & T_{F^cF^c}^\tau \end{pmatrix} \begin{pmatrix} \lambda_F^\tau \\ \lambda_{F^c}^\tau \end{pmatrix}$$

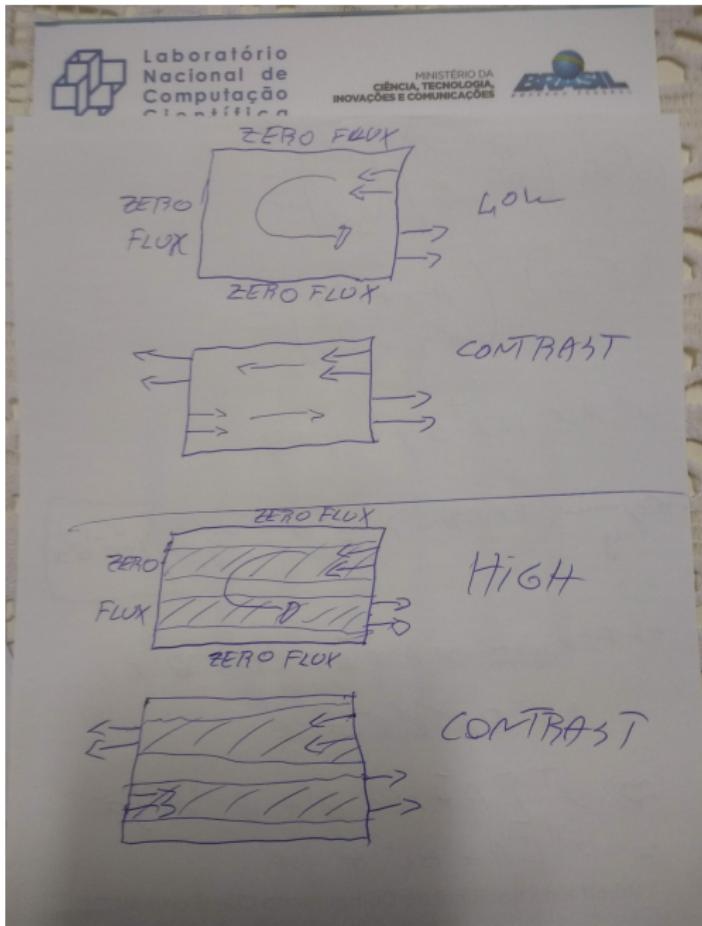
- ▶ Find eigenpairs $(\alpha_i^F, \tilde{\mu}_{h,i}^F) \in (\mathbb{R}, \tilde{\Lambda}_h^F)$ such that

$$(\tilde{\nu}_h^F, (T_{FF}^\tau + T_{FF}^{\tau'}) \tilde{\mu}_{h,i}^F)_F = \alpha_i^F (\tilde{\nu}_h^F, (\hat{T}_{FF}^\tau + \hat{T}_{FF}^{\tau'}) \tilde{\mu}_{h,i}^F)_F$$

for all $\tilde{\nu}_h^F \in \tilde{\Lambda}_h^F$ (\hat{T}_{FF}^τ is the Schur complement)

- ▶ $1 \leq \alpha_1^F \leq \alpha_2^F \leq \alpha_3^F \leq \dots$

Low/high energy flux extensions



Spectral Decomposition

- ▶ The user choose $\alpha_{\text{stab}} \geq 1$
- ▶ Decompose $\tilde{\Lambda}_h^F := \tilde{\Lambda}_h^{F,\Delta} \oplus \tilde{\Lambda}_h^{F,\Pi}$ such that

$$\tilde{\Lambda}_h^{F,\Delta} := \text{span}\{\tilde{\mu}_{h,i}^F : \alpha_i^F < \alpha_{\text{stab}}\},$$

$$\tilde{\Lambda}_h^{F,\Pi} := \text{span}\{\tilde{\mu}_{h,i}^F : \alpha_i^F \geq \alpha_{\text{stab}}\}.$$

- ▶ Define

$$\tilde{\Lambda}_h^\Pi = \{\tilde{\mu}_h \in \tilde{\Lambda}_h^f : \tilde{\mu}_h|_F \in \tilde{\Lambda}_h^{F,\Pi} \text{ for all } F \in \partial\mathcal{T}_H\},$$

$$\tilde{\Lambda}_h^\Delta = \{\tilde{\mu}_h \in \tilde{\Lambda}_h^f : \tilde{\mu}_h|_F \in \tilde{\Lambda}_h^{F,\Delta} \text{ for all } F \in \partial\mathcal{T}_H\}.$$

Decompositions

Decompose: $\Lambda_h = \Lambda_h^{\text{const}} \oplus \tilde{\Lambda}_h^{0,\Pi} \oplus \tilde{\Lambda}_h^\Delta$, where

- ▶ Λ_h^{const} - edge-wise constant fluxes (small dim.)
- ▶ $\tilde{\Lambda}_h^\Pi$ - “high energy” fluxes (slow decay)
- ▶ $\tilde{\Lambda}_h^\Delta$ - “low energy” fluxes (fast decay)

Define orthogonal projection: $P^\Delta T : \Lambda_h \rightarrow \tilde{\Lambda}_h^\Delta$

- ▶ If λ_h is local then $\mu_h^\Delta = P^\Delta T \lambda_h$ decays exponentially:

$$|T\tilde{\mu}_h^\Delta|_{H_A^1(\mathcal{T}_H \setminus \mathcal{T}_{j+1}(K))}^2 \leq e^{-\frac{[(j+1)/2]}{1+d^2\alpha_{\text{stab}}}} |T\tilde{\mu}_h^\Delta|_{H_A^1(\mathcal{T}_H)}^2,$$

- ▶ def. $P^{\Delta,j} T$ on patches & $\tilde{\lambda}_h^{\Delta,j} = -P^{\Delta,j}(T\lambda^{\text{const}} + T\tilde{\lambda}^{0,\Pi,j})$
- ▶ get a method with size $\sim \# \text{edges}$
- ▶ Mixed LOD: $\alpha_{\text{stab}} = \gamma \kappa \max_{\tau \in \mathcal{T}_H} (a_{\max}^\tau / a_{\min}^\tau) (1 + \log(H/h))^2$
- ▶ MHM: $\alpha_{\text{stab}} < 1$

Error

In energy norm:

$$|u_h - u_h^{\text{LSD},j}|_{H_A^1(\mathcal{T}_H)}^2 \leq c j^{2d} d^4 \alpha_{\text{stab}}^2 e^{-\frac{[(j-3)/2]}{1+d^2\alpha_{\text{stab}}}} (|T\lambda_h|_{H_A^1(\mathcal{T}_H)}^2 + |\tilde{T}g|_{H_A^1(\mathcal{T}_H)}^2)$$

Comments

- ▶ if patch size j grows, error $\rightarrow 0$
- ▶ faster decay with smaller α_{stab}
- ▶ needs minimum regularity

Oscillatory Problem (contrast= 1000)

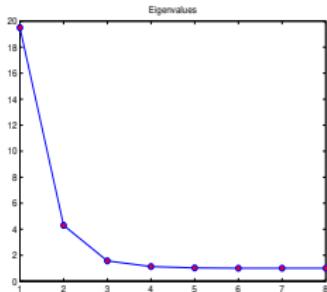


Figure: Plot of the eigenvalues.

	LOD
J=3	0.0092
J=2	0.0499
J=1	0.0778

	$\alpha_{\text{stab}} = 1.1$	$\alpha_{\text{stab}} = 3.0$
LSD (J=3)	5.7080e-11	2.7741e-06
LSD (J=2)	3.2054e-06	7.0387e-04
LSD (J=1)	0.0032	0.0291

Table: Relative energy error for LOD and LSD with different localizations

High-contrast case

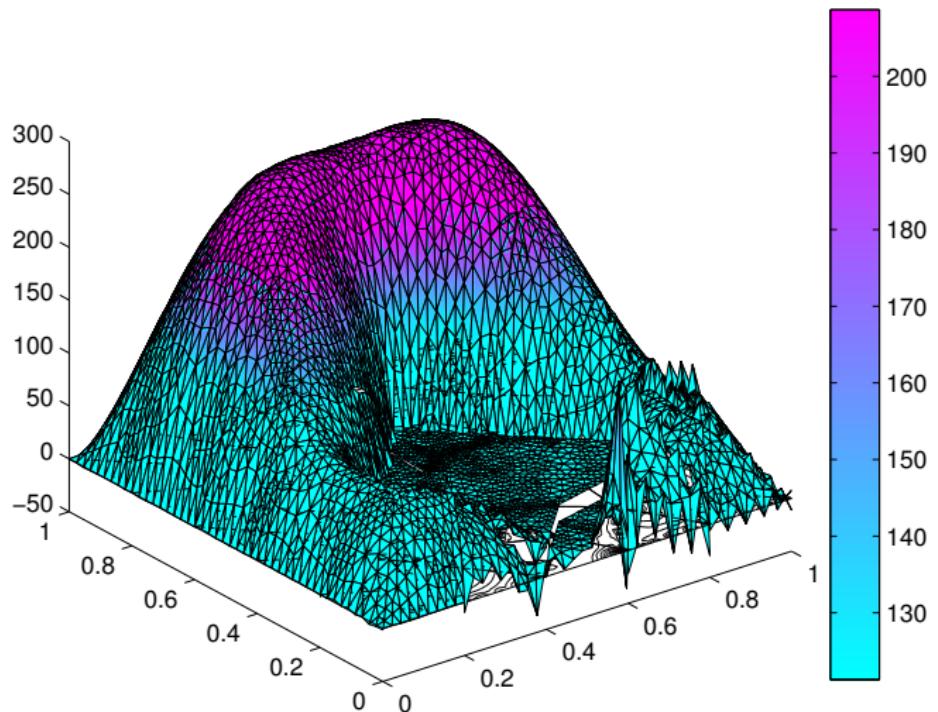


Figure: LSD solution with $\alpha_{stab} = 2$ and one patch

High-contrast case

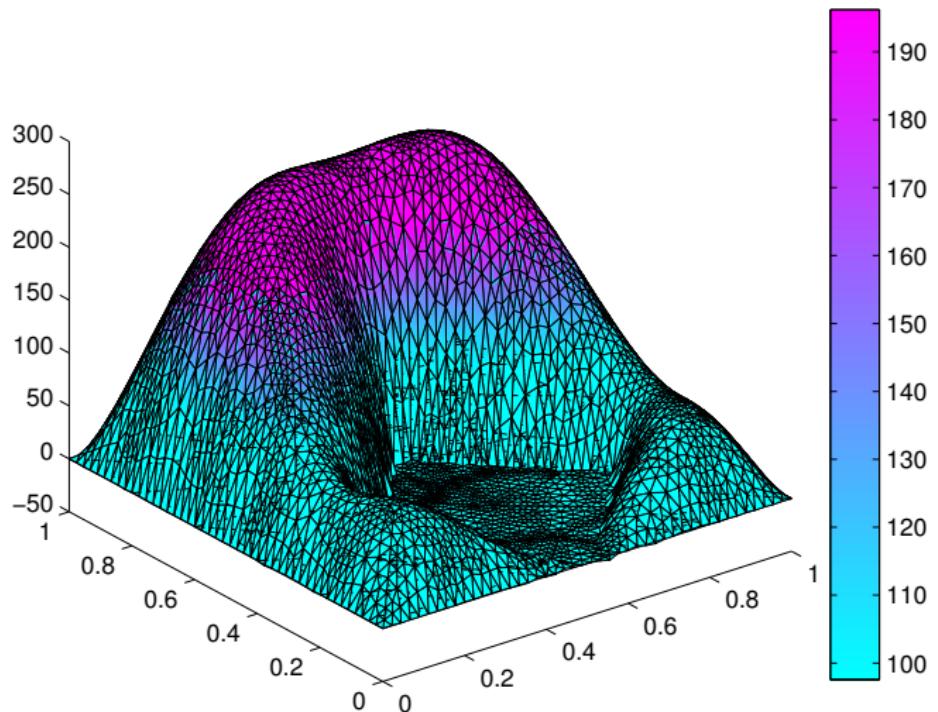


Figure: LSD solution with $\alpha_{stab} = 1.2$ and one patch

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Final remarks

- ▶ method based on primal hybrid formulation
- ▶ needs minimum regularity
- ▶ discrete flux decomposition
- ▶ exponential decay allows localization of global problems
- ▶ the choice of spaces depend on an appropriate spectral decomposition

References

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Localization of elliptic multiscale problems,
Math. Comp., 2014.
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Multiscale mixed finite elements,
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See LOTS of references in

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Hybrid Localized Spectral Decomposition for multiscale
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ArXiv, 2017.
-  M.; Sarkis, M.
Adaptive ACMS: A robust localized Approximated
Component Mode Synthesis Method
ArXiv, 2017.

Obrigado!!